

$$\sigma^* = f(\sigma^*) \Rightarrow \sigma_i^* = f_i(\sigma^*)$$

$$\Rightarrow \sigma_i^* = \frac{g_i(\sigma^*)}{\sum_{a \in A_i} g_i(\sigma^*)(a)}$$

$$\Rightarrow \sigma_i^* = \frac{1}{C} (\sigma_i^* + \text{Gain}_i(\sigma^*, \cdot))$$

$$\Rightarrow C\sigma_i^* = \sigma_i^* + \text{Gain}_i(\sigma^*, \cdot)$$

$$\Rightarrow (C-1)\sigma_i^* = \text{Gain}_i(\sigma^*, \cdot)$$

$$\Rightarrow \sigma_i^* = \left(\frac{1}{C-1} \right) \text{Gain}_i(\sigma^*, \cdot)$$

GAME THEORY

AND

THE ART OF WAR

Dean Dudley & Tony Martin

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Game Theory and the Art of War

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First published by the West Point Press, West Point, NY, in 2026.

Printed in the United States of America.

ISBN: 978-1-959631-26-2 (paperback) ISBN: 978-1-959631-27-9 (ebook)

LCCN: 2025939840

Cover design: Liliana Guia

Book design: Think Book Works

Cover images: iStock.com/sbayram (background), iStock.com/clu (illustration), iStock.com/duncan1890 (battle plan), iStock.com/AlyonaJitnaya (topographic map)

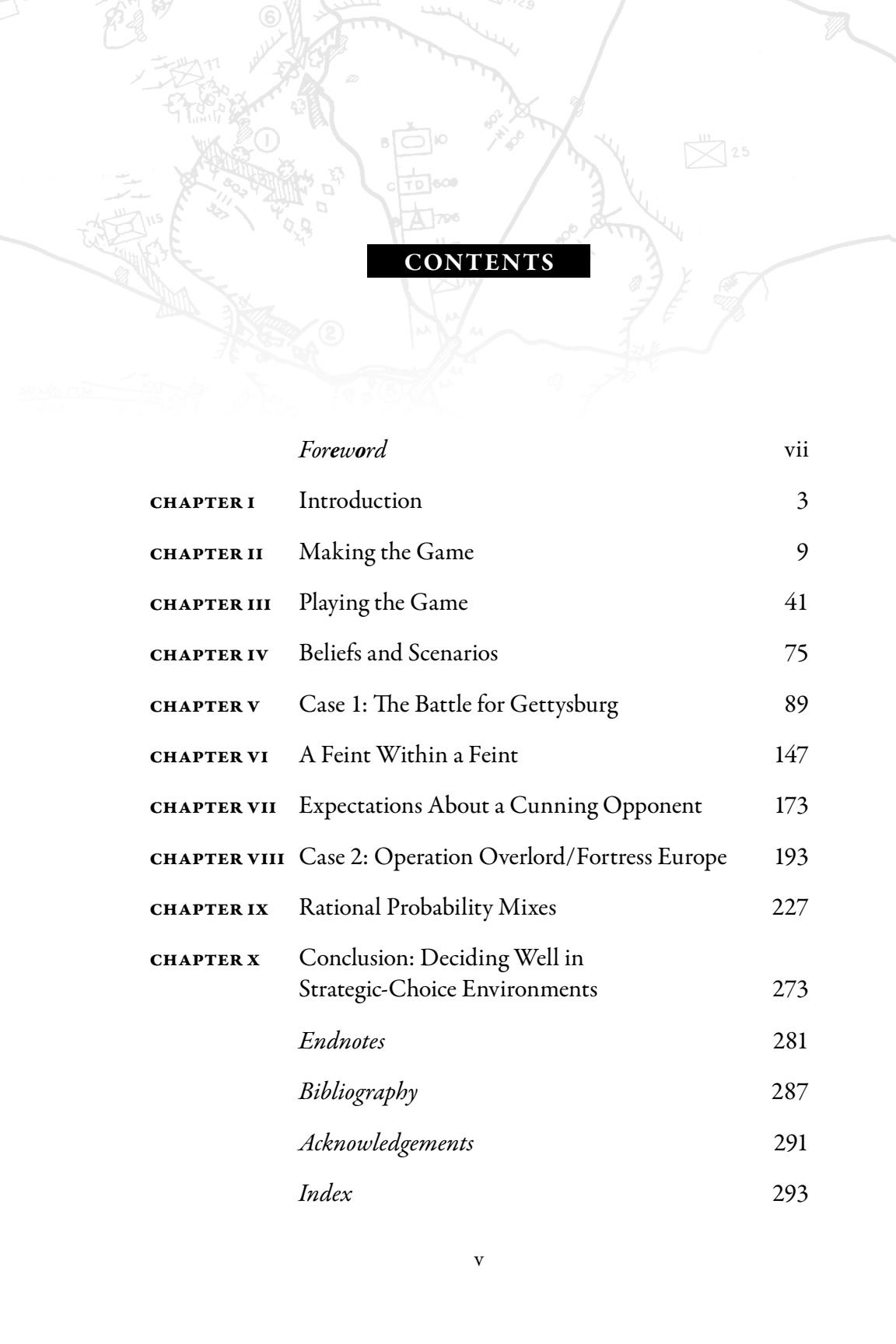
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FOREWORD

Every generation of leaders faces a different form of uncertainty. The tools change, the terrain shifts, and the pace of complexity accelerates—but the essence of leadership endures: the responsibility to make sound decisions under conditions of risk, competition, and consequence.

This book grows from that responsibility. Game theory, when translated from the classroom to the field, becomes more than a mathematical abstraction—it becomes a discipline of thinking. Game theory is the systematic investigation into individual decision-making in an environment where each decision-maker's choices affect their own outcomes and the outcomes of the others. This analysis allows the investigator to understand and anticipate how individual actors' decisions lead to specific outcomes. It teaches leaders how to see through the fog of uncertainty, to anticipate rather than react, and to navigate not only the adversary's intent but one's own biases and blind spots.

This project was a long time in the making. In the late 1990s, we became aware that the Department of the Army had no formal game theory education in any of its officer schools at any level. Seeing this deficiency in strategic analysis, we petitioned the Department of Social Sciences at the United States Military Academy (USMA) for permission to pilot a course in game theory focused on military strategic engagements. It was this course that spawned *Game Theory and the Art of War*.

We developed the fundamental game theoretic thought process and applied it to historical battles to analyze the decision making that emerged in these environments. The cadets then worked in small

teams to analyze a particular historical battle they had studied in their History of the Military Art classes, which they all took in their junior (cow) year. They were instructed to define the game that underlays the battle they had chosen, to apply the game theoretic thought process that was developed throughout the semester, and to analyze the principal players' course-of-action choices. This project coordinated the cadets' skills in historical analysis, leadership analysis, political science analysis, and mathematical and economic modeling. The final projects were inspiring to both of us.

Because of the cadets' astonishing ability to model sophisticated battle environments into truly poignant analysis, we thought a book sharing this process would be relevant to military decision makers, subordinate officers, senior noncommissioned officers, and those with a casual interest in military analysis. We designed *Game Theory and the Art of War* to provide the necessary game theoretic tools without the formal mathematical content present in most game theory texts. The point of this project is to make game theory accessible in a way that does not bog down the reader in formal mathematics but rather leads the reader through the game theoretic process using dialogue and simple game processes. We start by defining the fundamental pieces of a game, then build simple games, then apply equilibrium analysis to evaluate the games. From there we slowly layer on complexity to the point where the reader can analyze moderately complex games.

Game Theory and the Art of War provides the reader with the skills and knowledge necessary to apply substantive strategic analysis and move into the more mathematical texts on game theory if they choose to deepen their knowledge.

As our military and civilian leaders prepare for the demands of future conflict across domains, technologies, and political landscapes, developing this capacity for strategic reasoning will be as essential as developing new capabilities. In an age defined by complexity, clarity becomes a form of strength. Game theory provides that clarity, reminding us that the true contest of strategy is not simply to win, but to decide well when the stakes are highest.

GAME THEORY
AND
THE ART OF WAR

CHAPTER I

INTRODUCTION

*Warfare is the greatest affair of state, the basis of life
and death, the Way [Tao] to survival or extinction.*

It must be thoroughly pondered and analyzed.

SUN TZU, *ART OF WAR*

The driver lowers himself into the cockpit. He can see the water-like shimmer as the sun's light is distorted through the heated air on the surface of the salt flat. He tugs compulsively at his safety harness as he gets last-minute instructions from his crew chief over his headset. He hits the ignition, the jet engines whine, and all the panel lights show green. Years of research, engineering, and dedication led up to this moment. The timer signals, throttles are full, and distances compress. The land-speed record falls, or it does not. There is victory, and there is defeat.

A sprinter slowly walks onto the track and positions herself into lane four. She does a few last-minute stretches to work the nervous energy out of her ankles, calves, and thighs. She shakes her arms. She methodically sets her feet into the blocks and inches her thumbs up to the starting line. The starter calls "set," and eight sprinters tighten like watch springs. Years of training, practice, and conditioning layered onto rare talent and mental toughness led up to this moment. The pistol fires, strategy is applied, and energy is released. Less than ten seconds later there is victory, and there is defeat.

The center picks up the puck at center ice. He moves his head left but dumps the puck right to his wing. One defenseman is now out of position. As they cross the blue line, they face only one defenseman and the goalie. Years of training, practice, and conditioning layered onto rare talent and mental toughness led up to this moment. Play and counterplay are considered. The puck moves. Strategy is applied, and energy is released. In less than a heartbeat, there is victory and there is defeat.

Three events, three brief moments in time. While they all seem to be so similar, each is fundamentally different from the others. The driver is in competition with those who came before him, but his race is only against the clock. The clock is constant, consistent, and

mechanical. It does not change. In particular, the clock does not change in response to the choices made by the driver. There is no interaction between the driver's course of action and the way the clock ticks off minutes. There is no interplay between the driver and the length of a mile. The driver and the engineers are in control of their environment and therefore their collective fate.

The sprinter's situation is different from the driver's; she is in competition with seven others, and what they do affects the outcome. The sprinter's choice of action—whether she chooses to steadily accelerate through the race or whether she chooses to burst out of the blocks and maintain—interacts with similar decisions made in the other lanes. Each of the sprinters chooses a course of action in anticipation of the choices made by the others. Each sprinter's outcome is determined by the interaction of eight simultaneous decisions. None of the sprinters is master of her fate.

The situation faced by the wing and center is yet again different. They are in competition with the defenseman and goalie. What the defenseman and goalie do affects the outcome. Whether the wing feeds his center or keeps the puck depends on what he anticipates the defenseman and the goalie will do. What the defenseman and goalie do depends on what they anticipate the wing and center will do. In addition, there is time to implement a new course of action in response to a course of action chosen by the other side. One side will act and the other will react and a chain of actions will unfold. The sequence of actions taken by both sides will interact to determine the final outcome to determine the victor and the defeated. The key here is the interaction. Neither team is master of its own fate.

This book will concentrate on environments that are similar to the situations faced by the latter two examples: the sprinter, and the center and the wing. These are decision environments where the courses of action chosen by the participants interact to determine each participant's ultimate outcome. As it turns out, there are many types of human interaction that share the fundamental structure of these environments. In the economy, there are businesses competing in price or

share or quality in an interactive product market. Owners, managers, and workers in the workplace choose how to apply their efforts in a production process. In the political arena, there are candidates competing for votes in an election, legislators in session trying to construct legislation, and nations negotiating treaties. In the military, there are two generals on the battlefield trying to leave the field victorious. Any moment in which rational people consider courses of action in an environment where their choices interact with the choices of others to determine the ultimate outcome brings them into the arena of strategic choice and the realm of game theory.

Game theory is the rigorous and methodical examination of strategic choice. It is useful to decision-makers because it can systematically explain the merit of a course of action by an intuitive leader whose penetrating eye sees to the heart of a strategic-choice arena and chooses a course of action that leads to success through gut and instinct. It can also evaluate the potential success or failure of competing courses of action by a methodical leader who concentrates on doctrine and experience. All in all, game theory serves the decision-maker as an explanatory and evaluative tool. Both uses of game theory are powerful tools for any military decision-maker.

The explanatory power of game theory in strategic-choice arenas becomes even more important when we move away from the oversimplistic characterization of the battlefield as a game of chess, where the decision-maker moves nonintelligent and noncognizant military units as pawns and knights across the board. If a decision-maker has human subordinates, it is possible that the intuitive course of action is seen by a subordinate and not by the decision-maker. Under these circumstances, it is incumbent on the subordinate to clearly present and persuade the decision-maker of the merit of the subordinate's intuitive course of action. Here, the framework of game theory reveals its power as an explanatory tool, as a superstructure upon which the skin of assumptions, of actors, of courses of action, and of interactive outcomes, can be hung. In short, game theory can be used to systematically explain the relevance of a particular course of action.

The second benefit of game theory is its evaluative use. In planning sessions, game theory can be used to lay out all the relevant courses of action available to actors in a strategic-choice arena. The chosen courses of action come together in that arena to form interactive outcomes. These outcomes then have some value to all the actors involved. Decision-makers with foresight will look ahead to the outcomes and evaluate which courses of action will give them the most favorable outcomes. The game-theoretic structure is an excellent tool for systematically evaluating various courses of action with their associated interactive outcomes.

This book will concentrate on the military use of game theory. It is designed to help decision-makers use game theory as an explanatory or evaluative tool. In particular, it will focus on game theory's battlefield uses. First, the book will lay out the basic structure of the game-theoretic thought process. We will define what a game is, explain what the basic pieces are, and describe how they come together as an analytical tool. Next, we will develop techniques for analyzing specific game structures. Given a game, what is its reasonable resolution? Finally, we will apply that thought process to several historic battles to illustrate how the tool can be used.

This book will avoid much of the formal mathematics that is a common feature of academic game theory books. Rather, it will concentrate on narrative, the intuition behind the process and cases, to develop the game theory structure. Overall, we feel the reader will come away from this book with a very firm grasp of what game theory can and cannot do for decision-makers in strategic-choice environments. The reader will not only become a capable user of the game-theoretic thought process but also be able to use game theory to evaluate and explain courses of action in a strategic-choice environment.

CHAPTER II

MAKING THE GAME

In general, commanding a large number is like commanding a few. It is a question of dividing up the numbers. Fighting with a large number is like fighting with a few. It is a question of configuration and designation.

SUN TZU, *ART OF WAR*

To reap the benefits of the game-theoretic structure, it is important to know its parts and how those parts come together. There are three very basic components to a game. They are the players, the courses of action that the players can engage in, and the outcomes each player realizes when the game is over. The better each of the component parts is defined, the more accurately the game will describe the strategic-choice environment it represents.

Balanced against an accurate accounting of the environment is the need for tractability. By its very nature, game theory is an abstraction from reality. This abstraction is useful because it ignores superfluous details. It presents only the truly relevant details and allows the minds of those analyzing the strategic-choice environment to see into the heart of that environment and avoid being trapped or distracted by side issues. Game theory, when used properly, requires a balance between two competing goals: accurate description and tractability.

As recalled by William Bennet in *The Book of Virtues* (1993), in which he references the 1485 Battle of Bosworth Field, recounted in the poem “For the Want of a Horseshoe Nail,” the old folk wisdom states:

For want of a nail, a shoe was lost,
For want of a shoe, a horse was lost,
For want of a horse, a battle was lost,
For want of a battle, a kingdom was lost,
And all for the want of a horseshoe nail.¹

This poem implies that no level of detail is too trivial to be left out or ignored. It is true that in operationalizing any military action the details must be worked out, but in the planning stage, these details can prove to be impediments to seeing the greater strategic vision.

This is a basic conflict in all endeavors that model reality. The accuracy with which the model represents the strategic-choice environment and the ease with which the model can be used are in tension. Whether accuracy or tractability is accentuated depends upon the purpose of the model. To illustrate this point, consider a map.

The First Parable of the Map A friend and I decided to go on a road trip and explore the mid-Hudson River Valley region in New York State. We were both new to West Point *and* wanted to familiarize ourselves with the area. There are many sites of historical, cultural, geological, and just plain interest in the region. We had a regional road map, a tourist map, and a set of Metzger charts—high-definition terrain charts—to use as navigation tools. All three of these maps contained some level of description of the region.

The Metzger charts were a set of thirty charts in a binder. They had contour lines for every ten meters of elevation. They included all reported roads, trails, and firebreaks. They showed the position of all major structures, including power-distribution towers. They indicated whether the land was forested, meadowed, or marshy. They included the courses of rivers, creeks, and streams as well as the usual footprints of lakes and ponds. These charts were replete with information and were an exceptional physical description of the mid-Hudson River Valley.

The area road map was terser in its presentation of the region. It was a single sheet that folded out to be quite large. It included all roads and political boundaries, villages, and cities. The streets of the major villages and cities were represented on the back. It included large parks, lakes, and rivers. There was a lot of information available on the road map, but it was clearly a less detailed representation of the region than the Metzger charts.

The tourist map was at best a caricature of the region. It was a trifolded sheet of glossy paper. It only included the major interstate highways and the roads connecting the points of interest. The only villages and cities included were those with a major tourist attraction.

The tourist map was clearly a less detailed description of the region than either the road map or the Metzger charts.

As it turned out, the tourist map was the most useful for our road trip. We were able to circle through several interesting places, including General George Washington's headquarters, one of the oldest operating wineries in the United States, and the last encampment of the Continental Army. Though both the Metzger charts and the road map offered more detail and were therefore more realistic representations of the mid-Hudson River Valley, the detail they offered was not relevant to the task at hand. In fact, the level of detail was distracting and would have made navigating the road trip more difficult.

The Second Parable of the Map The next weekend, my friend and I planned to get out of our houses and do some camping and fishing. Friday night we pulled out the same three maps we used for our road trip to plan our fishing expedition. As the tourist map had served us well on the previous trip, we consulted it first. The lack of detail that made the tourist map attractive for our road trip made the map useless for planning our fishing trip. We needed to know where streams and lakes were located. We needed to know the trails and firebreaks that would make access to the streams and lakes easiest. We needed to find flat, dry, and lightly covered potential camping spots near our selected fishing spots. In fact, the level of detail that made the Metzger charts confusing and irritating on the road trip was exactly the level of detail we needed to successfully plan the fishing campout.

The lesson learned from the maps is that the accuracy of the representation of the strategic-choice environment in the game-theoretic model is dependent upon the intended use of the model. If you wish to understand the strategic level of, for example, the sweep of armies across Europe, you do not want to define your strategic-choice environment to the level of choice by individual soldiers. If you are planning an extraction raid—a military incursion into an enemy position to extract a person of interest—detail at the soldier level may be

appropriate. Game theory is flexible enough to allow both levels of abstraction and any level of abstraction in between.

I. Finding the Pieces

The lesson from the maps is that, before looking for the pieces of the game, you must first determine the level of generality or detail necessary for the specific task at hand. Is this a road trip, or is it a fishing expedition? Do you need to build a tourist map, or do you need to draw Metzger charts? Once you know the depth of the analysis, you can start assembling the pieces of the game, including, in particular, whose decisions and actions will have a significant impact on you in your strategic-choice arena. These decision-makers are the players in your game. After determining the players, you must ask what significant decisions can be made by both you and your opponent in the strategic-choice arena. These are the courses of action available in your game. Your next question then follows: How do your courses of action and the other players' courses of action interact to generate individual payoffs? These are the outcomes of your game. The fundamental structure of a game is illustrated in Figure 2.1. Decision-makers choose courses of action that interact in the strategic-choice environment to yield individual-level outcomes.

To help illustrate the thought process and technique of finding and assembling the pieces to a game-theoretic model, this section will follow the construction of two games. The first game will be a model of D-Day, and it will be called *Operation Overlord/Fortress Europe*. This is a game in which the participants are antagonistic. The second will be a model of the defense appropriations bill in the U.S. Senate, and it will be called *Craft a Bill*. This is a game in which the participants are not antagonistic but try to jointly devise a bill that best represents their political goals.

The Structure of a Game

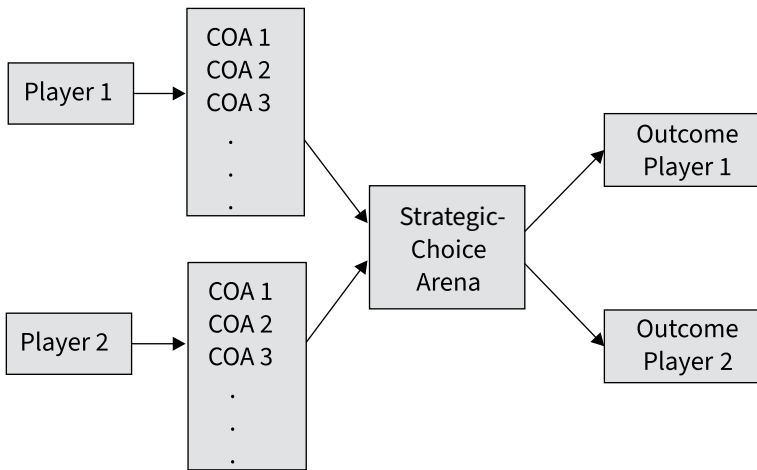


Figure 2.1: In a game, the players each choose their courses of action (COA). Those courses of action interact in the strategic-choice arena to determine each player's outcome.

A. The Players

As a youth I was a Minnesota Vikings football fan, and I remember seeing late-season games played at home. It was perversely satisfying to see a team like the Los Angeles Rams in a late November or December home game. Snow was swirling down on the field. Under these weather conditions, teams that relied on a pass offense were at a severe disadvantage. The weather conditions had as much an impact on the outcome of the game as the teams themselves. Often the sports announcers would refer to the weather as if it were another team on the field.

In this scenario, there are two different types of players in the game. The first is the type of players we readily recognize as human or strategic players. They are easily identifiable as the Vikings and the Rams. These are the players who make interactive choices. The second type of player in this scenario is not a strategic decision-maker but acts either randomly or outside of the strategic-choice arena. In this case it is the weather. Both the Vikings and the Rams, as they practiced the

week before the game, set their game strategies and focused on specific plays, expecting the weather to have an impact on the game. As the game unfolds, the efficacy of any particular play changes according to the intensity of the weather. Both teams change their offensive and defensive plays to compensate for the weather conditions. Here, nature is playing an active, though nonstrategic role in the game.

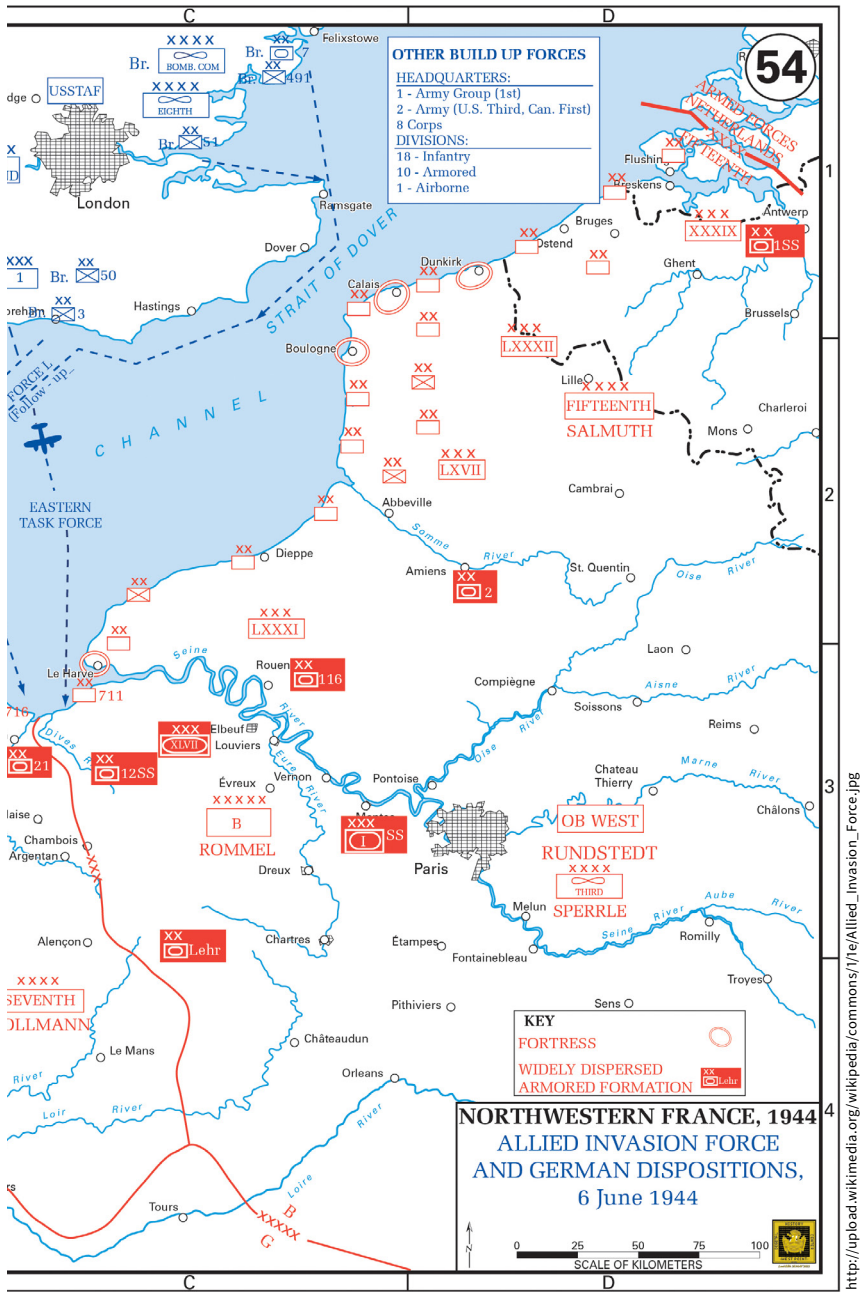
Just because nature can shape the play of a game does not imply it is always a player. Sun Tzu, in his fifth-century BC military treatise, the *Art of War*, recognized the importance of nature in the game of war. He informs generals that they must consider the nine terrains. Sun Tzu considers geography to be critical in determining the effectiveness of one military action over another.² Thus, geography is important in initial planning, but the terrain does not change as a battle unfolds to make a previously successful course of action useless. Here, nature is playing a passive role in the game of war. The difference between the weather in the football scenario and the terrain in Sun Tzu's battles is whether nature is an active force in or a passive part of the game. If nature is active, then nature is a player. If nature is passive, then it is not a player.

There may be human participants in a game who have active roles but are nonstrategic players. The referees, judges, and officials in most sporting events are examples. They are the arbiters of the rules. Like the weather, they have a random characteristic to them. Is that low inside fastball always a strike? Was that a forward-pass attempt or a fumble? When officials are making "judgment calls," are the officials always consistent? In addition to the randomness associated with close judgment decisions, these individuals can choose the intensity with which to enforce the rules. If the rules to a game are enforced rigorously, a team put together with a rough-and-tumble style may find itself at a disadvantage, whereas a team constructed for fineness may find itself at an advantage. The point to take from here is the existence of nonstrategic human and nonhuman players. When you list the players in a game, rounding up the usual suspects is easy. It is the *unusual* suspects you need to keep an eye out for.

The First Game: *Operation Overlord/Fortress Europe* The 1944 Allied invasion of Europe at Normandy depicted on the map in Image 2.1 is an excellent example of a very complex operation that can be represented very simply at the strategic level using game theory. In this game, the question of whether the Allies will invade Europe is moot. Both the Germans and the Allies know the invasion is imminent. So who are the decision-makers in this game?

The most obvious decision-makers, or the usual suspects, are both the German and Allied High Commands. These are the two groups who will lay out the basic structure of the invasion of Europe on the part of the Allies and the defense of Europe on the part of the Germans. It is important to understand that these two groups do not exist in a vacuum. Both rely heavily on the technical expertise of their theater commands. These are the people with local knowledge of the terrain as well as the troop capabilities. In addition to technical war-fighting expertise, there are political concerns as well. They must coordinate and placate multinational as well as internal political activities and rivalries.

In addition to the opposing high commands, theater commanders, and major political leaders, there are two other players in an amphibious landing and coastal defense. Weather, as a nonstrategic player, affects the efficacy of air cover, amphibious landing craft, and artillery. In addition, there is a randomness to battle outcomes often called the fog of war. Thus, *Operation Overlord/Fortress Europe* has a set of strategic players, depending upon the chosen level of abstraction, that include the Allied High Command, Allied theater commanders like General Dwight D. Eisenhower and General Benard Montgomery, Allied political leaders like Winston Churchill and Franklin D. Roosevelt, the German High Command, German theater commanders like Erwin Rommel and Gerd von Rundstedt, and finally the German political leader, Adolf Hitler. There were two nonstrategic players, the weather and the fog of war. Together these are the players in the game *Operation Overlord/Fortress Europe*.



The Second Game: *Craft a Bill* In the second game, consider the crafting of the defense appropriations bill in the U.S. Senate. This model, which stresses the importance of nonstrategic players in a game, will start by focusing on the strategic players. That is, it is time to round up the usual suspects. To do so, consider the game *Craft a Bill* as played by an imaginary politician, whom we will call Senator Foghorn.

Senator Foghorn is very pro-defense. The defense sector employs more voters than any other industry in his state. Senator Foghorn is crafting the Senate version of the defense appropriations bill. The senator wants to craft a bill that will pass through the Senate and become a law. Who are the players in this game of crafting a bill?

The first and easiest players to identify are the one hundred voting members of the Senate and the vice president, acting in his or her role as president of the U.S. Senate, if the vote is expected to be close. These are the people who, in the end, will decide whether the Senate bill will go to the president to be signed into law. Since the president must eventually sign the bill into law, the administration has some leverage in the bill-crafting process. Senator Foghorn's bill must address the concerns of the president. These are all of the direct players in the game of crafting a bill.

In addition, there are indirect players, people who can influence the game's eventual outcome, but who do not have an official role in the bill-to-law process. In a presidential administration, it is important to understand that the executive branch's defense policy does not evolve in a vacuum. At a minimum, the members of the president's security council and each of the service secretaries and chiefs have access to the president. Each of the services has a wish list of projects it wants funded. Though they do not vote, they are the expert members of the administration whose views are taken seriously. Many of these same people will be called upon formally in Senate hearings and informally in Senate offices to share their expertise with the legislative branch.

There are also those outside of both governmental branches who can play a part in this game. These are the leaders in the defense-industrial

base. They are the leaders of the companies that sell weapon systems to the U.S. Department of Defense. They may or may not be brought in to testify at a Senate hearing, but they will most certainly visit Senate and administration offices. In addition, there are groups that would like to see defense appropriations take a smaller portion of the national budget. If defense spending is smaller, there is more of the national budget available for other concerns. All these players together are called special interest groups.

From this list of potential players, it becomes obvious that the game *Craft a Bill* can be very complex. There is interaction among the services, between the political parties, between the executive and legislative branches of government, and finally among industries and interest groups competing for federal dollars. In addition to this myriad of players, there are a lot of ways each of these players can choose to interact with one another. Each player's potential courses of action are legion.

B. Their Courses of Action

Once each of the players in a game has been determined, it is time to decide what each is capable of doing. In what courses of action can any and all players engage to affect the outcome of the game? This is not a list of most likely courses of action but a catalogue of all courses of action.

Courses of action come in two basic types. The first type is discrete actions. These are like the actions that a car's driver can make at an intersection. The driver can continue to go straight, turn left, or turn right. With discrete courses of action, the decision-maker has a limited menu of options to choose from. Discrete courses of action are the easiest to catalogue. The second type is continuous actions. These are like the course corrections a ship captain can make. The captain can adjust course one degree, ten degrees, thirty degrees, or any fraction of a degree in between. With continuous courses of action, the decision-maker has a spectrum of options to choose from. By their very nature, continuous courses of action are very difficult to catalogue.

Returning to *Operation Overlord/Fortress Europe* If the analysis of this game starts in early 1944, both the Allies and Germany know that an assault on Europe from England is imminent.³ As the attacking force, the Allies get to choose when, where, and how large the assault will be. The Germans get to choose how to defend against and repel the assault.

Consider the Allies' potential courses of action. Since the equipment needed to mount an amphibious assault on Europe must be accumulated, the decision of when and how large are most likely the same. The earlier the assault, the less available equipment, and the smaller the attack force. Conversely, the later the assault, the more available equipment, and the larger the attack force. Regardless of whether these two courses of action are truly one, they are continuous in nature. That is, dates and size can be chosen from a spectrum. This is not true of where a course of action can take place. The northern coastline of continental Europe is continuous, but the entire coast is not suitable for landing an invasion. There are very few suitable beaches. The Allies had a menu of just two reasonable choices: the Bay de Seine in Normandy, and Pas de Calais.

The Germans had to defend Western Europe while fighting a major front on their east in Russia. They had to determine how many resources to devote to each theater. Given the resource decision, the Germans then had to determine how many and where to install fixed coastal defenses, as well as how many and where to place mobile defenses. The fixed-coastal-defense decision is continuous. Even though Normandy and Calais were the logical invasion sites, the resource balance in fixed coastal defenses allocated between the two is a decision along a spectrum. Does Germany build the coastal defenses evenly at each site, or is one built up faster than the other? If one is built up faster, how much faster? The placement choice of mobile defenses, however, is discrete in nature. Does Germany decide to slow the invasion forces with its coastal defenses and prepare its mobile forces (infantry and armor) to repel the invasion from inland,

or do mobile forces get placed up front with the fixed coastal defenses to deny the Allies a foothold?

Returning to *Craft a Bill* In Senator Foghorn's theoretical game, there are many players. There are the senators, the administration, and the special interest groups. In politics there are many courses of action open to the players of the political game. They range from the mundane to the immoral. For the sake of simplicity, the players will only have access to the mundane courses of action.

The senators can vote for the bill, vote against the bill, or amend the bill. The act of amending the bill is a continuous course of action along many domains, even though the amendment is either approved or disapproved. For example, a senator can propose an amendment to change the distribution of funding across programs in the bill, to add new or delete current programs, or to increase or decrease the total funding.

The administration can sign the bill into law, not sign the bill and let it become law, or can veto the bill. These are discrete actions.

The special interest groups can influence the voting of senators and the decision of the administration through campaign contributions and direct constituent mailings. The size of the campaign contribution is a continuous course of action, ranging from nothing to the budget of the interest group. Constituent mailing is a discrete choice. The interest group either does it or does not.

Of course, left out of this description of courses of action for the *Craft a Bill* game is the backroom part of the political game. None of the promises of one vote here for another vote there, none of the "we will bury your project in committee if you don't support this," and none of the arm-twisting is put in this example. Though the omitted courses of action are valid and probably more realistic representations of the political process, the goal here is to show how to construct a game, not how to create the most realistic example.

C. Their Outcomes

We know the players, and we know their courses of action. These are the last pieces of information required to assemble the game and their outcomes. What does each player have after each of their chosen courses of action interact? These are the outcomes reaped by each player. To evaluate the efficacy of various courses of action, the set of all outcomes associated with all possible courses of action must be listed.

To generate the list of potential outcomes, a complex game of “what if” must be played. What happens if I choose my course of action 1 and my opponent chooses her course of action 3? What is my outcome, and what is his? What if I choose course of action 4 and she chooses course of action 2? This thought process must proceed until every possible combination of interacting courses of action is exhausted and every possible set of outcomes is determined.

Ordinal and Cardinal Values Once the outcomes are determined, they must be valued. There are two basic ways outcomes can be valued: They can have ordinal values or cardinal values. Ordinal values are numerical values that represent the relative preference rankings of the outcomes. Ordinal values are those we assign when we play the game “on a scale of 1 to 10,” where “10” is meant to represent the ultimate value and 1 represents the worst. The only information these numbers impart is the order of preference. For example, suppose a serving of brussels sprouts that has an ordinal ranking of 1 is less preferred to a serving of broccoli that has an ordinal ranking of 2. Though the number 2 is twice the number 1, since the ranking of preference is ordinal in nature, a serving of broccoli does not necessarily give twice the value of a serving of brussels sprouts. In addition, a serving of brussels sprouts and a serving of broccoli need not yield a preference ranking of 3. Ordinal rankings only relate to the order of preference.

There are outcomes that have cardinal values. If the game played is a war of attrition, then the outcome of a battle may be measured in body count. Body count is a cardinal measure. Two hundred dead is

twice one hundred dead, and in a war of attrition that makes a difference. In addition, an outcome that yields one hundred dead coupled with an outcome that yields two hundred dead does yield three hundred dead overall. A cardinal value represents more than just the order of preference of a particular outcome; it also represents the magnitude of that outcome and can be used to make comparisons between different outcomes. With cardinal valuations, one can determine whether an outcome is half, twice, or any multiple of another. With cardinal valuations, the adding of two outcomes also makes sense.

The Parable of the Crosswalk To illustrate the process of determining and valuing all possible outcomes, consider an intersection of two roads. A pedestrian and a car traveling in the same direction reach the intersection simultaneously. The pedestrian is going to cross the intersection and continue in the direction she is currently heading. The driver is going to make a left-hand turn that will cross the pedestrian's path. For simplicity, each player has two courses of action. The pedestrian can cross the intersection without yielding to the driver, or the pedestrian can yield to the driver, then cross the intersection. The driver can turn without yielding to the pedestrian, or the driver can yield to the pedestrian, then continue.

To determine the potential outcomes, we must consider all possible choice pairs made by the driver and the pedestrian. The choice-pairs column in Table 2.1 delineates all four possible choice pairs available in the game. So for the first-choice pair, in which both the pedestrian and the driver yield to the other, if there is no further iteration of the game, then the driver and the pedestrian will both sit indefinitely at the intersection waiting for the other to proceed. The second-choice pair, in which the pedestrian yields and the driver turns without yielding, the driver goes through the intersection first, followed by the pedestrian. Yielding the outcome, the driver continues first, followed by the pedestrian crossing.

Table 2.1: Choice and Outcome Pairs

	CHOICE PAIRS	OUTCOME PAIRS
1	P: Yield Then Cross D: Yield Then Continue	P: Wait Indefinitely D: Wait Indefinitely
2	P: Yield Then Cross D: Continue Without Yielding	P: Cross Second D: Continue First
3	P: Cross Without Yielding D: Yield Then Continue	P: Cross First D: Continue Second
4	P: Cross Without Yielding D: Continue Without Yielding	P: Hit by Driver D: Hits the Pedestrian

This thought process continues until all the choice pairs are coupled with outcome pairs. Once all the possible outcomes are determined, valuations can be applied to each player’s potential outcome. These valuations can be as simple as ordinal rankings, thus representing the player’s preference order. That is, for example, the pedestrian prefers crossing first to crossing second and prefers crossing second to waiting indefinitely and prefers waiting indefinitely to being hit. So the most preferred outcome would be assigned the highest-ranking value, the next preferred a somewhat lower value, and so on until the least preferred outcome is assigned the lowest value. A similar process could be applied to the driver’s outcomes.

If a scale of 1 to 4 is used, with 4 representing the ranking of the most preferred outcome and 1 the ranking of the least preferred, then the outcomes can be assigned values as in Table 2.2.

The valuations assigned to each outcome could also be cardinal values. If, for example, the measure of an outcome’s valuation was represented by the time lost at the intersection, or the income lost due to the time spent at the intersection, then the game would have cardinal valuations.

Table 2.2: Choice Outcome and Value Pairs

	CHOICE PAIRS	OUTCOME PAIRS	VALUE
1	P: Yield Then Cross D: Yield Then Continue	P: Wait Indefinitely D: Wait Indefinitely	P: 2 D: 2
2	P: Yield Then Cross D: Continue Without Yield	P: Cross Second D: Continue First	P: 3 D: 4
3	P: Cross Without Yielding D: Yield Then Continue	P: Cross First D: Continue Second	P: 4 D: 3
4	P: Cross Without Yielding D: Continue Without Yield	P: Hit by Driver D: Hits the Pedestrian	P: 1 D: 1

Continuing Play in *Operation Overlord/Fortress Europe* For *Operation Overlord/Fortress Europe*, though the game itself is quite complex, there are only a few possible outcomes: the Germans repel the Allied invasion; the Allies establish a defensible landing area; and/or the Allies break through the German defenses and move inland. The Germans would assign a higher preference valuation of 3 to their most preferred outcome, down to a preference valuation of 1 for their least preferred. The Allies could rank the outcomes in exactly the opposite order.

It is possible that the Germans and the Allies have cardinal measures of the valuation of the outcomes. For example, there could be a strategic assessment of the probability the realized outcome leads to a particular side's preferred resolution of the war. Thus, if the Germans repel the landing, the Allies have a low probability of defeating Germany, say 10 percent. If the Allies establish a defensible landing area, their probability of defeating Germany increases to 70 percent. Finally, if the Allies break through the German defenses, their probability of defeating Germany rises to 90 percent.

Notice how much more information is inherent in the cardinal measure than in the ordinal measure. Not only is the preference order represented by the probability measure (90 percent is preferred to 70 percent, is preferred to 10 percent), but the differences between the probabilities represent the intensity of the preferences. The intensity of preferences is not conveyed in the ordinal valuations.

Continuing Play in *Craft a Bill* In *Craft a Bill*, there are three basic types of players: the administration, the Congress (the House of Representatives and the Senate), and the contractors. All three have specific types of outcomes associated with the game, but the game itself will end with a defense appropriations bill either passed as law or not passed as law.

The contractors' outcomes in the *Craft a Bill* game are the easiest to define. The contractors are playing for profits or at the very least expected profits. The reason any contractor would want its project included in the defense appropriations bill is for the greater profits the company would realize from more robust sales to the military.

As for the members of the presidential administration, this game assumes that they have some public policy agenda. That is, that the president comes into office to exercise a mandate from the people. So the administration gains satisfaction from achieving that mandate. The closer the enacted public policy is to the perceived mandate, the more satisfied the administration is with the outcome. More specifically, the administration has a most-preferred defense appropriations bill, and the closer the final bill is to that most-preferred bill, the more preferred the outcome is to the administration.

Finally, Congress wants to redistribute federal tax money back into their congressional districts or states. More federal money reallocated to their states translates into a higher probability of reelection. Each member of Congress wants to be reelected so he or she will craft a bill to increase his or her likelihood of reelection. The more money spent in their individual districts, the more preferred is the bill.

II. Assembling the Game

In addition to the fundamental building blocks of a strategic-choice environment, players, courses of action, and outcomes, there is the effect of time. How time enters into the game environment is very important. There are many ways time affects a strategic-choice environment. We will pay careful attention to two different time effects.

First, time can affect how the players are allowed to choose their courses of action. Is the strategic-choice environment turn based? That is, does one decision-maker choose a course of action, then the next, as in a game of chess, with each taking a turn until the end, when the outcomes are realized? This type of strategic-choice environment, in which the players take turns, is called a sequential-play game.

Alternatively, all the decision-makers may choose their courses of action and release them into the strategic-choice environment to interact and realize their respective outcomes all at once. Here is a strategic-choice environment like the game rock-paper-scissors, with each decision-maker revealing his or her course-of-action choice simultaneously. This environment is called a simultaneous-play game.

The second-time effect is on how often the game is played. Do the decision-makers come together and confront the strategic-choice arena once, like a major battle? This is a single-shot game. Or do they interact multiple times? This is a repeated-play game.

Right now, we will concentrate on the first type of time effect, sequential play, and then simultaneous-play games.

A. Taking Turns

In sequential-play games, the players take turns choosing and implementing courses of action. Choices made early in the game will influence the choices available to players deeper into the game. In the simplest of sequential-play games, the first player makes the first move. Player 1 chooses a course of action from the set of courses of action available. It is now Player 2's turn. Player 2 chooses a course

of action from the set available to Player 2. It is important to keep in mind that Player 2 makes choices contingent on the choices made by Player 1. So Player 1’s choice may affect the set of courses of action available to Player 2. Once the choices are made, they interact to determine the outcomes realized by each, Player 1 and Player 2.

A simple sequential game can be easily displayed as a decision tree, as in Figure 2.2. This is called the extensive form of a game. At the top of the decision tree is the player who initiates the game play. In this case, it is Player 1. Directly below and designated with arrows is the menu of courses of action available to Player 1. Once Player 1 chooses a particular course of action, game play moves to Player 2. The boxes labeled with the player are called decision nodes of the game and represent where players must make course-of-action choices. In this simple game, once Player 2 chooses a course of action from those available at their node, the game resolves to the realized outcomes.

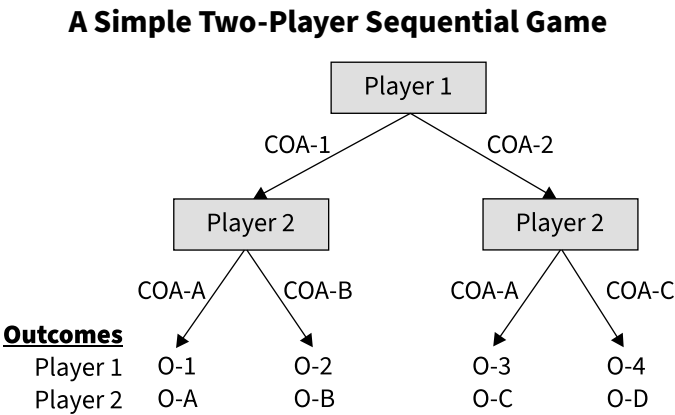


Figure 2.2: Player 1 chooses between courses of action 1 and 2. Once Player 1 chooses, Player 2 can choose between courses of action A and B or courses of action A and C. The game then resolves to the players’ respective outcomes (O), O-1 through O-4 for Player 1 and O-A through O-D for Player 2.

So, for example, if Player 1 from Figure 2.2, choosing between course-of-action 1 and course-of-action 2, chooses course-of-action 2. Then game play moves to Player 2, following the arrow labeled course-of-action 2 from the Player 1 decision node to Player 2's right-hand decision node. Here Player 2 must choose between course-of-action A and course-of-action C. If Player 2 chooses course-of-action C, then the game resolves, following the arrow labeled course-of-action C to the outcome where Player 1 receives outcome O-4 and Player 2 receives outcome O-D.

Taking Turns in *Operation Overlord/Fortress Europe* Now we get to put together a very simplified version of one of the games that we have been building throughout this chapter. To start, let us make the simplifying assumption that the Allies have already decided on an amphibious invasion across the English Channel and that the Germans know that this decision has been made. So the game starts with Germany determining its defense of Western Europe. It is very important to remember that Germany is fighting a two-front war. A lot of German war resources and personnel are being absorbed on the Russian front, even with this impending invasion of Europe.

As discussed in the section on courses of action earlier in this chapter, Germany has two distinct types of defensive units. These are the fixed coastal defenses and the mobile infantry and armor units. The density of buildup of the fixed coastal defenses at Calais and Normandy is a continuous decision. For simplicity's sake in this example, Germany can concentrate them in Calais and build up Normandy slower, concentrate them in Normandy and build up Calais slower, or build up Calais and Normandy evenly. The mobile forces could be held inland to respond to the invasion site or pre-positioned forward in Calais and Normandy. This gives the Germans six potential courses of action, which are also represented in Figure 2.3:

1. Concentrate fixed coastal defenses in Calais with mobile defenses inland.

- 2. Concentrate fixed coastal defenses in Calais with mobile defenses pre-positioned forward.
- 3. Concentrate fixed coastal defenses in Normandy with mobile defenses inland.
- 4. Concentrate fixed coastal defenses in Normandy with mobile defenses pre-positioned forward.
- 5. Build Normandy and Calais evenly with mobile defenses inland.
- 6. Build Normandy and Calais evenly with mobile defenses pre-positioned forward.

The Allies respond to the German defense and choose whether to land at Normandy or at Calais. Again, this is an oversimplification for the sake of example, since the Allies get to balance off the continuous choices of how built up the Germans are and how built up their forces are by choosing the invasion date. Given this simplification, there are six Allied decision nodes, one under each of the German courses of action. They all look like Figure 2.4, in which the Allies have two course-of-action choices:

- 1. Invade at Calais.
- 2. Invade at Normandy.

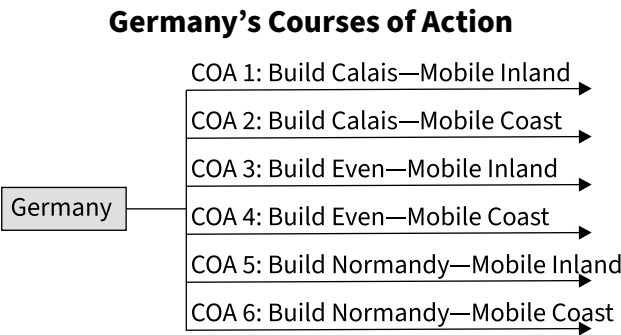


Figure 2.3: German defensive courses of action (COA) all include the paired fixed coastal defense and mobile infantry/armor defense choices. In this simplified version of the game, there are six possible combinations.

Allied Courses of Action

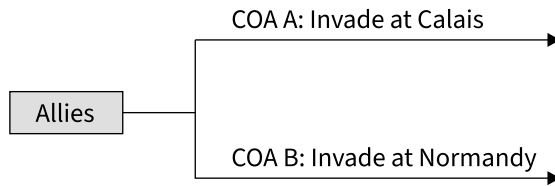


Figure 2.4: From the Allied decision node, the Allies can choose either COA A, invade at Calais, or COA B, invade at Normandy.

In the simple two-player game used to introduce sequential-play games, after Player 2 chose a course of action, the game resolved itself to the outcomes. This game is a little more complex. Once the Allies choose where to invade Europe, a battle will ensue. The outcome of a battle is not deterministic but probabilistic. That is, there is no guaranteed victor. In the end someone will have won, and someone will have lost, but the victor is not known before the battle. We may know who has the better chance of winning the battle, just like the oddsmakers know which football team is more likely to win a game. Nevertheless, either team may win.

In Greek mythology, the outcome of battle was in the hands of the gods, and the gods were capricious. So in the battle for Troy described in Homer's *The Iliad*, there were the Greeks and Trojans as primary combatants or players, but once the two sides met in battle, in came the gods as a third player. In the extensive form of the *Operation Overlord/Fortress Europe* game, we will deal with the uncertainty of battle by pretending a third player enters the game as the Germans and Allies come to battle. This third player is Nature.

When the Germans and Allies come together in battle, there are three significant possible outcomes: The Germans repel the Allies; the Allies gain a defensible foothold on the continent; or the Allies break through the German defenses and move inland. Regardless of German or Allied preparation and courses of action, all three of

these outcomes are possible. They are not all three equally likely given German and Allied course-of-action choices. For example, the Allies are more likely to gain a defensible foothold if the Germans keep their mobile defense inland than if the Germans have significant infantry and armor on the beach. By the same token, the Allies are more likely to be repelled if they invade at a point with highly concentrated fixed coastal defenses than one with less concentrated fixed coastal defenses.

Figure 2.5 represents the possible outcomes and their associated probabilities—P1, P2, and P3. Thus, the nonstrategic player, Nature, is solely a mechanism to couple outcomes and probabilities given the choices made by the strategic players, Germany and the Allies.

The probabilities P1, P2, and P3 can come from many sources. For example, if this were a board game in which dice were used to determine the outcome of the battle between the Allies and Germany, the probabilities would be determined by the dice. So rolling a one would mean the Allies are repelled, while a two, a three, or a four means the Allies gained a foothold on the beach, and a five or a six means the Allies break through the German defense. In this way, the Germans have a one-in-six or 16.67 percent chance of repelling the invasion. The Allies have a three-in-six or 50 percent chance of gaining a foothold and a two-in-six or 33.33 percent chance of breaking through. The true probabilities may be unknown, but from history or experience, an estimate or good guess of the probabilities can be made. If history tells us that a force like the Allies has a 50 percent chance of being defeated by a dug-in defense, a 40 percent chance of gaining a foothold, and a 10 percent chance of breaking through the German defense, then these estimates may be used for P1, P2, and P3.

On the technical side of probabilities, if the game is fully defined, the game must resolve itself to one of the outcomes. This means that the probabilities along a given node must add up to 100 percent. So $P1 + P2 + P3 = 100$ percent, as was the case in the two previous examples, in which 16.67 percent + 50 percent + 33.33 percent = 100 percent and 50 percent + 40 percent + 10 percent = 100 percent.

The Role of Nature in a Game

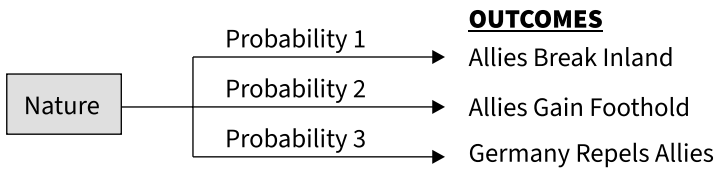


Figure 2.5: Nature is a third player in the game. Nature determines the probability of the potential outcomes given the courses of action chosen by Germany and the Allies. Those probabilities may be determined by history, military doctrine, or the physical environments.

Now that the pieces of the game are assembled and the extensive form representation is understood, the game tree for *Operation Overlord/Fortress Europe* is represented in Figure 2.6. To fill in the probabilities for the outcomes at each node, the long-accepted rule of thumb is that an attacking force with a three-to-one advantage in strength has a 50 percent chance of overcoming a well-fortified defense and a 50 percent chance of being repelled.^{4,5} A rough count of German and Allied strength, as displayed in Image 2.1, is as follows.⁶ The Allied forces have an overwhelming air and naval superiority and invade with six amphibious divisions and three airborne divisions. The Germans, along the channel coast, have 21 fixed divisions, seven mobile infantry divisions, six panzer-armor divisions, and two airborne divisions. The panzer divisions are equipped at about half-strength.^{7,8}

With this additional information, a quick assessment of outcome probabilities can be generated and the extensive form of the game *Operation Overlord/Fortress Europe* can be completed. Figure 2.6 gives a possible representation of the game.

Operation Overlord/Fortress Europe

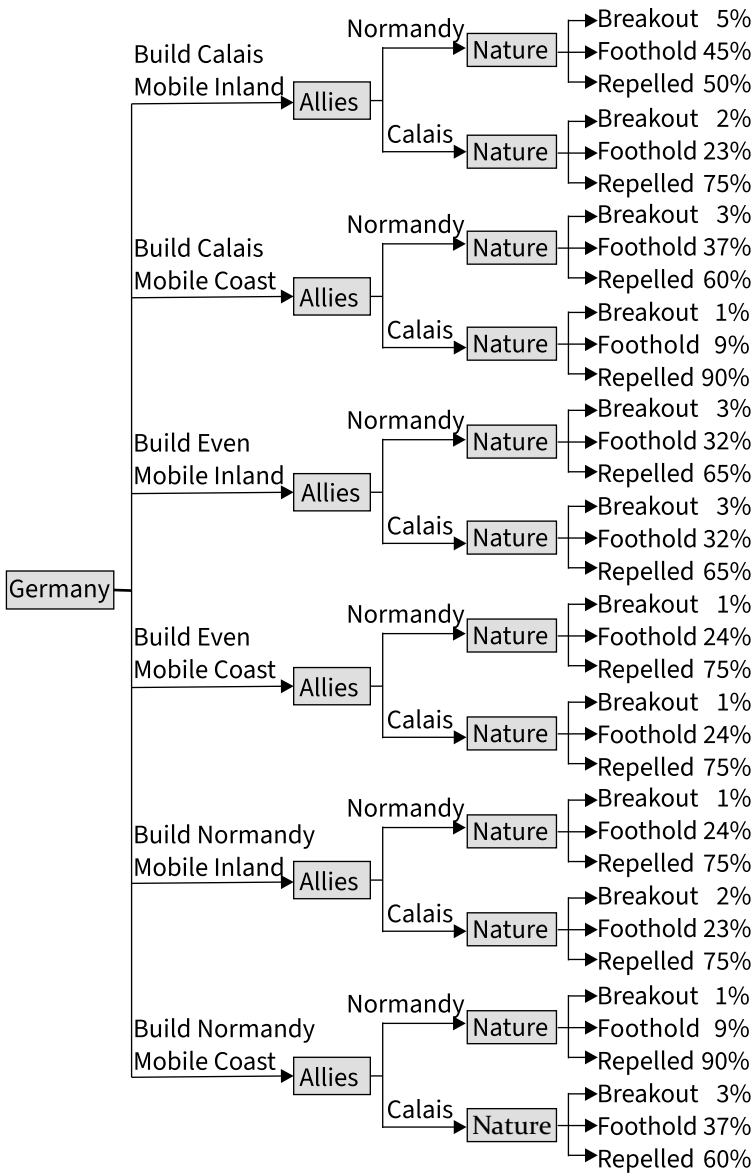


Figure 2.6

B. Ready, Set, GO!

In simultaneous-play games, the players choose and implement their courses of action together. Simultaneous-play games are represented differently from turn-based or sequential-play games, but the same basic information must be presented. That is, the players must be represented, their courses of action must be delineated, and the potential outcomes must be defined. Simple simultaneous-play games can be easily represented in the game-matrix format called the normal form of a game. A two-player game in which each player has two courses of action to choose from is the easiest simultaneous-play game to model.

Consider Figure 2.7, in which Player 1 can choose between COA 1 and COA 2 in a game with Player 2, who can choose between COA A and COA B. A play of the game consists of Player 1 and Player 2 each choosing, simultaneously, a course of action and having it resolve to the outcome stage. From the perspective of Player 1 in Figure 2.7, the outcome realized is contingent upon Player 1's and Player 2's courses of action. For example, if Player 1 chooses COA 2 and Player 2 chooses COA A, then the outcome realized by Player 1 is in the highlighted cell in Figure 2.7, where Player 1's course-of-action row and Player 2's course-of-action column intersect. In particular, Player 1's outcome is Outcome-3.

Player 1's Game		
		Player 2
		COA A COA B
Player 1	COA 1	Outcome-1 Outcome-2
	COA 2	Outcome-3 Outcome-4

Figure 2.7: Corresponding to each of Player 1's and Player 2's course-of-action (COA) choice pair is an outcome (O) realized by Player 1. That outcome is displayed in the cell at the intersection of Player 2's COA choice column and Player 1's COA choice row.

A similar representation can be made for Player 2’s perspective of the game. As represented in Figure 2.8, Player 1 and Player 2 each choose their respective courses of action, and the game resolves itself to Player 2’s outcomes. If the same course-of-action choices as presented in Figure 2.7 are chosen again, then the game for Player 2 resolves to outcome Outcome-C.

Player 2's Game

		Player 2	
		COA A	COA B
Player 1	COA 1	O-A	O-B
	COA 2	O-C	O-D

Figure 2.8: Corresponding to each of Player 1’s and Player 2’s course-of-action (COA) choice pair is an outcome (O) realized by Player 2. That outcome is displayed in the cell at the intersection of Player 2’s COA choice column and Player 1’s COA choice row.

To represent the whole game in normal form, the information in Figure 2.7 and Figure 2.8 is merged. Thus, the course-of-action choice pairs are associated with their respective outcome pairs. This is presented in Figure 2.9.

The Normal Form Game Assembled

		Player 2	
		COA A	COA B
Player 1	COA 1	O-1 O-A	O-2 O-B
	COA 2	O-3 O-C	O-4 O-D

Figure 2.9: In the normal form of the game, Player 1’s and Player 2’s outcome (O) pairs are matched into the cells associated with their interactive course-of-action (COA) pairs.

The normal form of the game can be used to model the simple game discussed in the Parable of the Crosswalk. In that scenario, a driver and a pedestrian make simultaneous decisions upon entering or yielding at an intersection. The players’ courses of action and the outcomes were presented in Table 2.2. The game is modeled in Figure 2.10. Instead of representing the actual outcome pairs associated with course-of-action choice pairs, the individual valuations associated with each outcome are represented.

The Crosswalk Game in Normal Form

		Driver	
		Yield Then Continue	Continue w/o Yielding
Pedestrian	Yield Then Cross	2 2	3 4
	Cross w/o Yielding	4 3	1 1

Figure 2.10: The normal form of the crosswalk game with the players’ valuations of their outcomes substituted for their actual outcomes.

Critical to analyzing any strategic-decision environment is building up the fundamental structure of the game that represents the environment of interest. This chapter focuses on determining the level of analysis, the players or decision-makers in the game, their respective course-of-action choices, and finally, their potential outcomes. The next chapter will build the concepts and analytical tools necessary to understand player behavior in the game that has been modelled. Thus, the strategist who wants to explain why certain course-of-action choices are desirable will have the tool set to do so, and the decision-maker who wants to know the best course-of-action choice will have the tool set to do so.

CHAPTER III

PLAYING THE GAME

Thus, it is said that one who knows the enemy and knows himself will not be endangered in a hundred engagements. One who does not know the enemy but knows himself will sometimes be victorious, sometimes meet with defeat. One who knows neither the enemy nor himself will invariably be defeated in every engagement.

SUN TZU, *ART OF WAR*

Now that the basic techniques of modeling a strategic-choice environment using game theory have been discussed, it is time to develop the tools to evaluate these games. To that end, we must ask: How would a reasonable decision-maker choose one course of action over another? Knowing that in a strategic-choice environment, outcomes are realized after the choices of the participants interact, reasonable decision-makers will make their choices anticipating the choices of the other participants. This thought process is the heart of strategic behavior: decision-making contingent upon the actual or anticipated behavior of others in the environment.

That said, it is necessary to make assumptions about the other players in the game. The first assumption is that of empathy. Overall, people are fundamentally more alike than different—the things that drive one person to act are the same ones that drive others.¹ This is the assumption that supports American psychologist Abraham Maslow's famed hierarchy of needs.² In his hierarchy, Maslow outlines an ordering of human motivational factors, from the basic needs of survival up to more esoteric human intellectual needs, like freedom, respect, self-esteem, and the desire to be all that one can be. The point here is that for us to look out at other people and anticipate or be comfortable with their behavior, we must also understand that the things motivating them to act have to be similar to those things motivating us to act. The idea that humans are similar enough that one person can, at an intellectual level, walk a mile in another person's shoes is the foundation of most social science.

In addition to empathy, we need to assume that people are fundamentally rational.³ To pursue game theory, the required level of individual rationality is not demanding. An individual is sufficiently rational for game theory if that individual is able to set consistent

preferences over potential outcomes and will seek to obtain more preferred outcomes over lesser-preferred outcomes. Consistent preferences require that an individual can determine whether one outcome is better, worse, or the same as another outcome. And, if there are three potential outcomes—win, draw, or lose—whereby the individual prefers a win to a draw and prefers a draw to a loss, the logic must hold that the individual prefers a win to a loss.

Given these assumed attributes—that is, a player is empathetic and rational—games will naturally gravitate toward equilibrium outcomes.⁴ Thus, there is a level of comprehensibility in game play.

I. Game Equilibrium

Given the assumption that players would choose more preferred outcomes over less preferred outcomes, it makes sense that players would choose courses of action that would yield them their most preferred outcomes. A player would be able to achieve this goal if there were no other players in the game. In a game with more than one player, the players must use empathy by choosing the courses of action that give them their most preferred outcomes as they anticipate the courses of action the other players in the game may choose.

To illustrate this point, consider a game of siege. In such a game, there is a walled city and a raiding army. This theoretical walled city can either send out its small city guard force to do battle with the raiders, or it can defend the city from behind its sealed gates and walls. The raiders can either avoid the city or lay siege to it. If a force of the city guard rushes out and the raiders lay siege, the city guard will be destroyed, and the raiders will loot the city unhindered. If the city guard rushes out to attack while the raiders avoid the city, both the city guard and the raiders will suffer losses from skirmishes with the guard, but the raiders will be able to do some limited looting. If the city guard locks down the city and the raiders avoid the city, the city and guard are unmolested, but the raiders get no loot. If the city guard

locks down the city and the raiders lay siege, neither the guard nor the raiders suffer losses, but the city's residents must ransom their freedom.

The city and the raiders, therefore, have a set of preferences over the potential outcomes. On the one hand, the city leaders and guard prefer being left unmolested to suffering some losses and limited looting; suffering some losses and limited looting to ransoming their freedom; or ransoming their freedom to unhindered looting. On the other hand, the raiders prefer unhindered looting to ransom; ransom to losses and limited looting; or losses and limited looting to no loot at all.

This game can be looked at as either a simultaneous or sequential-play game. That is, after the city's population and the raiders notice each other, each side picks a course of action simultaneously or one side chooses a course of action, then the other responds. First, let us examine a sequential representation of the game. In addition, let us assume the city gets to choose its course of action first. Thus, the city guard can see the raiders approaching from their towers and can select a course of action before the raiders know what the city has chosen. Thus, the raiders choose their course of action after they know the city's course-of-action choice.

The *Game of Siege* as a Sequential-Play Game In the extensive form of the sequential-play *Game of Siege*, the city chooses its course of action first followed by the raiders' course-of-action choice. The raiders' decision indicates they can choose their course of action with perfect knowledge of the city's prior course-of-action choice. Once both players have chosen their respective courses of action, the game resolves to reveal each player's outcomes. Using ordinal rankings from one to four for the city's preferences regarding its potential outcomes—with four representing the most preferred outcome down to one representing the least preferred outcome—the city's outcomes are valued. The same method is used for valuing the raiders' outcomes. Figure 3.1 represents the fully defined extensive form of the sequential-play siege game.

Before developing a mechanism for finding an equilibrium to this game, we will first consider ways to look at the game with a critical

eye. Are there things about the game that are illogical or irrational once the game has been defined? If these are addressed, how does the game as portrayed in Figure 3.1 transform?

Should it choose the course of action to attack, the outcome rankings available to the city are one if the raiders choose to siege and three if the raiders choose to avoid the city. The outcome rankings available to the city, should it choose the course of action to close its gates, are two if the raiders choose to siege and four if the raiders choose to avoid. The city's outcome rankings of one and three for the course-of-action attack are always less than the outcome rankings of two and four for the course-of-action close, regardless of the raiders' course-of-action choice. Thus, the city will always achieve a lower-valued outcome if its leaders choose the course of action to attack. Assuming the city's leadership is acting as a rational player—meaning a player seeking a more preferred outcome to a less preferred outcome—it would be irrational for the city to choose the course of action to attack in the game as portrayed in Figure 3.1. That is, the city's outcome from choosing the course of action to attack is dominated by some other course of action. In this simple game, we can say that the course of action to attack is dominated by the course of action to close the gates.

Using this same analytical technique when evaluating the course-of-action choices available to the raiders, we find that the raiders, when they choose the course of action to avoid the city, will achieve outcomes valued at one if the city chooses the course of action to close its gates and two if the city chooses the course of action to attack. Additionally, the raiders, when choosing the course of action to lay siege, will achieve outcomes valued at three if the city chooses the course of action to close its gates and four if the city chooses the course of action to attack. Since the raiders' outcome values are always lower if they choose the course of action to avoid the city, the raiders' course of action to avoid is dominated by their course of action to lay siege. Therefore, it would be irrational for the raiders to ever choose the course of action to avoid attacking the city.

The Sequential Game of Siege

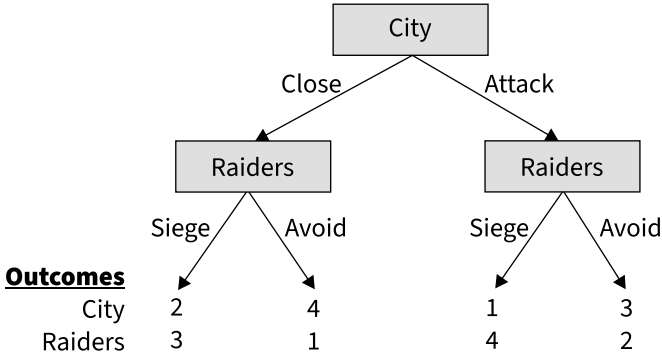


Figure 3.1: The extensive form of the sequential-play *Game of Siege*. The city chooses its course of action first, followed by the raiders’ course-of-action choice. The raiders have perfect knowledge of the city’s course-of-action choice. Once both players have chosen their respective courses of action, the game resolves to their outcomes.

Thus, the game portrayed in Figure 3.1 can be redrawn as the game portrayed in Figure 3.2. Since the city’s course-of-action choice to attack is dominated by its course-of-action choice to close the gates, the course of action to attack will never be chosen by a rational player. To indicate this, the course-of-action-to-attack branch is pruned out of the city’s rational-choice set and is grayed out in Figure 3.2. The exact same pruning mechanism is applied to the raiders’ decision-making process. Since the raiders’ course of action to avoid the city is dominated by their course of action to lay siege, the course of action to avoid the city is pruned from the raiders’ rational-choice set and is grayed out in Figure 3.2.

Following the pruned *Sequential Game of Siege* presented in Figure 3.2, the city leadership is first to choose its course of action. Having determined that it is never in its best interest to choose the course of action to attack, the city chooses its course of action to close its gates. Once the city leaders have made their course-of-action choice, the *Sequential Game of Siege* moves down, following the ray from the city’s decision node labeled “close” to the raiders’ corresponding

decision node just below, as shown in Figure 3.2. At this node, the raiders will choose between their courses of action to siege or to avoid. Again, having determined that it is never rational for the raiders to choose their course of action to avoid, the raiders will instead choose to siege the city. After the course-of-action choices are made, the *Sequential Game of Siege* resolves down to the outcomes in which the city realizes an outcome valuation of two and the raiders realize an outcome valuation of three.

That is, once the irrational course-of-action choices for the city and for the raiders are pruned from their respective choice sets, the *Game of Siege* reduces to the simple “choice” sequence of the city closing and the raiders laying siege. This path of play in the *Game of Siege* resolves to the outcomes of the city paying a ransom with no casualties to its guard, providing an outcome preference ranking of two, and the raiders receiving a ransom with no casualties, providing an outcome ranking of three.

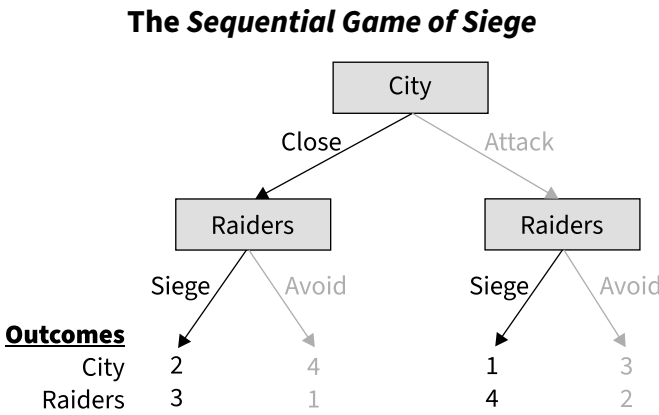


Figure 3.2: The city’s dominated course of action, “attack,” and the raiders’ dominated course of action, “avoid,” are pruned from the game. This yields the rational path of game play, whereby the city chooses its course of action, “close,” and the raiders choose their course of action, “siege,” yielding outcomes valued at two for the city and three for the raiders.

The *Game of Siege* as a Simultaneous-Play Game Now look at the siege game as a simultaneous-play game. With simultaneous play, the game can be represented in the normal form as previously developed in Figure 2.9. In this example, we will assume that neither player knows the other player’s course-of-action choice prior to making their own course-of-action choice. Thus, the city leaders and the raiders choose their respective courses of action simultaneously.

Keeping the same outcome structure and the same preference ordering as laid out in the *Sequential-play Game of Siege*, the normal form of the *Simultaneous-play Game of Siege* is as represented in Figure 3.3.

Normal Form of the *Game of Siege*

		Raiders	
		Siege	Avoid
City	Close-up	2 3	4 1
	Attack	1 4	3 2

Figure 3.3: The normal form of the *Simultaneous-play Game of Siege*. The city and the raiders choose their respective courses of action simultaneously without knowing the choice of the other. Once both players have chosen their respective courses of action, the game resolves to their outcomes.

Given the outcome structure of the siege game as defined in the previous section, and the preference ordering over those outcomes held by the city and the raiders, the *Simultaneous-play Game of Siege* has similar characteristics as the sequential version of the game. That is, the game will always deliver an outcome with a lower preference

ranking if the city chooses the course of action to attack. Given this outcome structure, it would be irrational for the city to choose its course of action to attack. This is illustrated in Figure 3.4 by graying out the city’s course-of-action attack row. Similarly, due to the outcome structure and preference ordering of the raiders, it would be irrational for them to choose their course of action to avoid. This is illustrated in Figure 3.4 by graying out the raiders’ course of action “avoid” column.

Pruning out the irrational course-of-action choices for the city and the raiders, the normal form of the *Simultaneous-play Game of Siege* reduces to the simple “choice” set of the city closing its gates and the raiders laying siege to the city. Thus, simultaneous play of the siege game resolves to the outcomes of the city paying a ransom, with no casualties to its guard, providing an outcome preference ranking of two and the raiders receiving a ransom, with no casualties, providing an outcome ranking of three, as shown in Figure 3.4.

Normal Form of the *Game of Siege*

		Raiders	
		Siege	Avoid
City	Close-up	2 3	4 1
	Attack	1 4	3 2

Figure 3.4: Pruning out the irrational courses of action in the normal form of the *Simultaneous-play Game of Siege*. The city will never choose its course of action to attack, and it will never choose its course of action to avoid, as indicated by being grayed out. This yields the rational game play, whereby the city chooses its course of action to close and the raiders choose their course of action to siege, yielding outcomes valued at two for the city and three for the raiders.

Using Figure 3.4 and looking at the game's rational-choice outcome (the city chooses close, and the raiders choose siege), the city cannot improve its outcome by unilaterally changing its course of action from closing its gates to attacking. That is, given the raiders' chosen course of action, the city chose the course of action that gave it its highest-ranked outcome. Or, to view it another way, the city choosing to close its gates is its best response to the raiders' choice to lay siege. This is also true for the raiders. The raiders cannot improve their outcome by unilaterally changing their course of action from laying siege to the city to avoiding it. That is, given the city's chosen course of action is to close its gates, the raiders chose the course of action that gave them their highest-ranked outcome. In game theory talk, the city chose its best response to the raiders' course-of-action choice, and the raiders chose their best response to the city's course-of-action choice. A game is in "Nash equilibrium"—so-named for the American mathematician John Nash, who originated some of game theory's central tenets—when each player has chosen their best response to the other players' course-of-action choices, and when no individual player can achieve a higher-valued outcome through a unilateral course-of-action change.⁵

The Game Parable of the Crosswalk Now let us look at the game having to do with crossing at a crosswalk. In this game, introduced in Figure 2.10 and reproduced as Figure 3.5, there are no irrational courses of action for either player that can be pruned from game play. If the pedestrian chooses to yield to an oncoming car and then cross, the pedestrian will realize an outcome with a valuation of two or three. If the pedestrian chooses to cross *without* yielding, the pedestrian will realize an outcome with a valuation of one or four. The best course of action for the pedestrian depends upon what the driver chooses to do. The situation is the same for the driver. There is no equilibrium in dominant courses of action.

The Crosswalk Game in Normal Form

		Driver	
		Yield Then Continue	Continue w/o Yielding
Pedestrian	Yield Then Cross	2 2	3 4
	Cross w/o Yielding	4 3	1 1

Figure 3.5: In the normal form of the crosswalk game, neither player has a course-of-action choice that is always inferior.

To find the equilibriums in this game, each of the outcome cells must be examined to see if they meet the equilibrium requirement. That is, none of the players can unilaterally improve their outcome by changing their course-of-action choice. Start in the northwest outcome cell, in which both the driver and the pedestrian have chosen their course of action to yield, as illustrated in Figure 3.6, where, in this game, both players have the outcome of an eternal wait. Ask yourself: Does either player have unilateral incentive to change their course-of-action choice?

Start with the driver. If the driver unilaterally changed his or her course of action from “yield” to the course of action “continue without yielding,” the game would no longer resolve to the northwest outcome cell but would resolve to the northeast cell instead. The driver’s outcome valuation has, through a unilateral course-of-action choice change, improved from two to four. This means that the northwest outcome cell cannot represent an equilibrium cell. At least one player, the driver, can improve his or her outcome valuation through a unilateral course-of-action change.

The Crosswalk Game in Normal Form

		Driver	
		Yield Then Continue	Continue w/o Yielding
Pedestrian	Yield Then Cross	2 ↓ 4	3 → 4
	Cross w/o Yielding	4 ↓ 3	1 ↓ 1

Figure 3.6: Examining the northwest cell of this game shows that both players have unilateral incentive to choose an alternate course of action; thus, the northwest cell does not meet the criteria of being a Nash equilibrium to the game.

Still considering the northwest outcome cell, notice that the pedestrian’s outcome valuation can also improve through a unilateral course-of-action choice change from yielding to crossing without yielding. This unilateral course-of-action change causes the game to move from the northwest cell to the southwest cell. The pedestrian increases their outcome valuation to four instead of two.

The existence of an outcome-enhancing unilateral course-of-action choice change for the driver is signified by the arrow pointing from the driver’s outcome, valued at two in the northwest cell, to the driver’s outcome, valued at four in the northeast cell, as indicated in Figure 3.6. Similarly, the downward-pointing arrow indicates that, for the pedestrian, a unilateral outcome valuation increase is possible.

It is easily seen that both players can also improve their outcomes from the southeast cell through a unilateral course-of-action change. This indicates that neither the northwest outcome cell nor the southeast outcome cell represents equilibrium outcomes.

Focusing on the southwest cell in Figure 3.7, in which the pedestrian crosses and the driver yields, neither the pedestrian nor the driver can improve their outcomes through a unilateral course-of-action change. That is, if the pedestrian knew in advance that the driver would yield, the pedestrian’s best course-of-action choice is to cross.

Similarly, if the driver knew in advance that the pedestrian would cross, the driver’s best course-of-action choice is to yield. Therefore, the course-of-action choices made by the pedestrian and the driver are mutual best responses to the other’s course-of-action choice.

Notice that neither player can realize an outcome improvement through unilateral course-of-action change from the northeast outcome cell in Figure 3.7 as well. Thus, both the southwest and the northeast cells represent equilibrium outcomes. The crosswalk game has two equilibria; the driver yields and the pedestrian crosses, resulting in a value of three to the driver and four to the pedestrian, and the pedestrian yields while the driver continues, resulting in a value of three to the pedestrian and four to the driver.

Looking at Figure 3.7, we notice that cells with arrows pointing out are not equilibrium outcomes. Cells with arrows only pointing in are equilibrium outcomes. These cells meet the criteria for a Nash equilibrium. That is, each player’s course-of-action choice is a best response to the course-of-action choices of the other players. Or no player can improve their outcome valuation by unilaterally changing their course-of-action choice, given the course-of-action choices of the other players.

The Crosswalk Game in Normal Form

		Driver	
		Yield Then Continue	Continue w/o Yielding
Pedestrian	Yield Then Cross	2 ↓	3 ↑
	Cross w/o Yielding	4 ↓	1 ↑

Figure 3.7: Outcome cells with arrows pointing out are not equilibrium outcomes. Outcome cells with arrows only pointing in are equilibrium outcomes. These cells meet the criteria for a Nash equilibrium.

II. Look Forward, Reason Back

Sequential-play games, represented in the extensive form, have a more restrictive equilibrium concept called sub-game perfect equilibrium.⁶ To be a sub-game perfect Nash equilibrium of a sequential-play game, the players' course-of-action choices in each of the game's sub-games must themselves be Nash equilibria to each of the game's sub-games. In a sub-game perfect equilibrium, each player's decision sequence contains only *credible* course-of-action choices. It eliminates potential equilibria that rely on players making *noncredible* threats.

A sub-game perfect equilibrium is found using a solution technique called backward induction. Backward induction requires the analyst to look forward into the game tree to the decision nodes that are farthest out and reason course-of-action choices back up the tree. The outcomes that survive the climb back up the tree are the sub-game perfect equilibrium outcomes.

To illustrate the technique of backward induction, consider the sequential-play form of the siege game we discussed in the previous section and represented in Figure 3.1. This game was designed to have an intuitively obvious dominant course-of-action equilibrium with the outcome of the city paying a ransom to end the siege.

The first step in determining the sub-game perfect equilibrium and its associated outcomes is to start at the outcomes themselves and move up the decision tree to the first decision nodes above the outcomes. In the extensive form of the siege game, there are two such nodes, and both are the raiders' decision nodes. In Figure 3.8, they are labeled Sub-game 1 and Sub-game 2.

Sub-game 1 represents the decision the raiders must make if the city chooses to close its gates. At this point the raiders can choose to mount a siege of the city, giving them an outcome valued at three, or they can choose to go around and avoid the city, giving them an outcome valued at one. Under these circumstances, the raiders would choose to mount a siege and earn the higher-valued outcome. That is, the raiders' best response to the city choosing to close its gates is for the raiders to choose the course of action to lay siege.

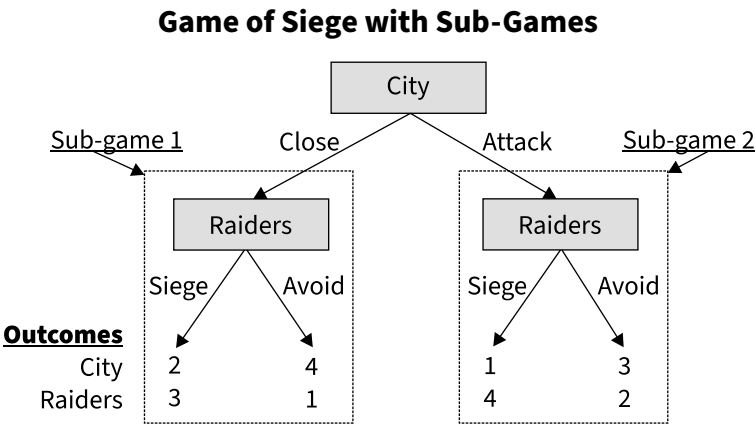


Figure 3.8: Starting from the outcomes and working up to the first decision node, the game is broken into its first sub-games.

Similarly, in Sub-game 2, in which the city has chosen the course of action to attack, the raiders can choose to mount a siege, yielding an outcome valued at four, or they can go around and avoid the city, yielding them an outcome valued at two. Again, the raiders would choose the course of action to lay siege to earn the higher-valued outcome. The raiders would choose the course of action to lay siege as their best response to the city’s course-of-action choice to attack.

In Figure 3.9, the raiders’ best responses in both sub-games are represented in black, whereas the other courses of action are grayed.

With the raiders’ sub-games resolved, as seen in Figure 3.9, backward induction can be applied to the associated outcomes, which can then move up a level in the game tree. Thus, it is reasonable for the city’s leaders to expect the raiders to choose the course of action to lay siege if they choose to close their gates, and it is reasonable for the city to expect the raiders to choose the course of action to lay siege if the city chooses to attack. The city can choose the course of action to close its gates, yielding an outcome valued at two, or the city can choose the course of action to attack, yielding an outcome valued at one. If the city leaders act rationally, their best response to the raiders’ rational course-of-action choices is to choose the course of action to close the gates and earn the outcome valued at two as indicated in Figure 3.10.

Game of Siege with Sub-Games Resolved

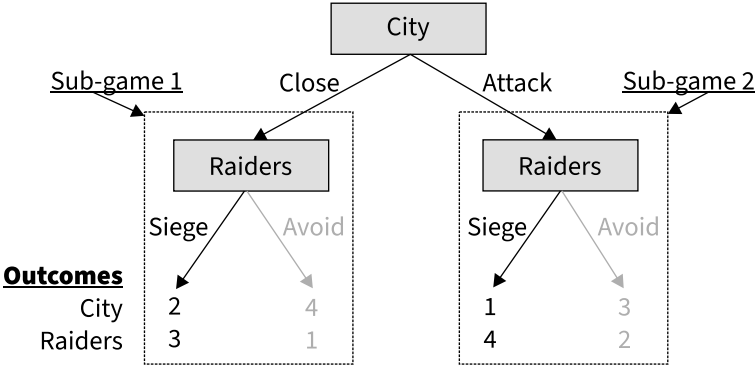


Figure 3.9: The raiders choose the course of action that is the best response to the city’s course-of-action choice in each of the sub-games. In this case, the course of action to siege yields the highest-valued outcomes in both sub-games. The raiders’ best responses are indicated by being left highlighted, while the other irrational course-of-action choices are pruned, as indicated by being grayed out.

Game Resolved up One Level
Through Backward Induction

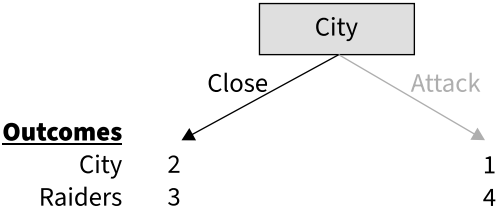


Figure 3.10: Backward induction resolves a game up level by level until there are no more levels. The game can now be solved for its sub-game perfect equilibrium outcome.

Backward induction takes the rational resolution of the farthest-out sub-games and pulls the outcomes up a level. At this point, a new game tree can be constructed taking the players’ best-response choices into account. This new game tree may itself have a set of sub-games that can be resolved up yet another level. The backward-induction process continues until the game is transformed into a form as shown

in Figure 3.10, where the game is reduced to a final sub-game. At this point, the game can be solved for the sub-game perfect equilibriums.

The initial extensive form of the game can be reassembled, as in Figure 3.11, by following the backward-induced decision paths of each of the players in each of the respective sub-games. The decision paths that survived the backward-induction process are the game’s sub-game perfect equilibriums. These equilibriums represent the courses of action that rational players would follow and the outcomes they would receive. In the *Game of Siege*, the sub-game perfect equilibrium is the one in which the city will choose the course of action to close and the raiders choose their contingent strategy to lay siege to the city’s closed gates and the course of action to lay siege if the city chooses to attack. Although for brevity and in casual conversation, the sub-game perfect equilibrium to the *Game of Siege* is loosely defined as the city chooses to close and then the raiders choose to siege. If there were a set of city strategists determining a strategy for the *Game of Siege* before the interaction were to occur, they would weigh the merits of choosing the course of action to close or to attack depending on their knowledge

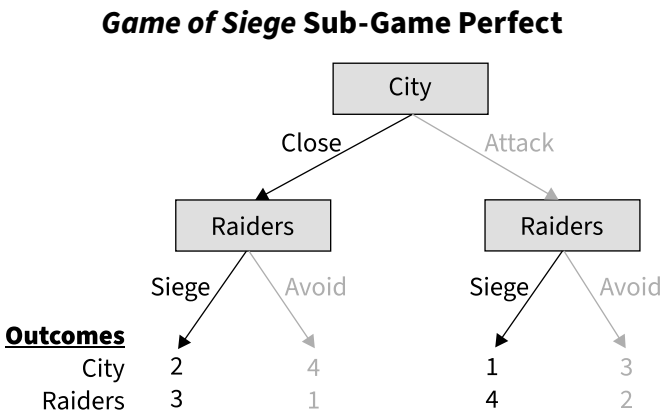


Figure 3.11: In sub-game perfect equilibrium, the city chooses to close and the raiders choose to siege, with the outcome valued at two for the city and three for the raiders.

of the raiders. At the same time, the raiders' strategists would be weighing the merits of four contingent courses of action: to lay siege whether the city chooses to close its gates or attack; to lay siege if the city chooses to close and to avoid if the city chooses to attack; to avoid whether the city chooses to close the gates or if the city chooses to attack; and to avoid if the city leaders choose to close or choose to lay siege should the city choose to attack.

Sequential Games with Random Nonstrategic Players Finding a sub-game perfect equilibrium in a game with random nonstrategic players such as the forces of Nature is a little more complicated than in the siege game previously presented. Nature acts in a probabilistic manner. That is, the outcome due to Nature's action is determined randomly, according to a known probability distribution. As it turns out, there is a way to attach an expected value to random outcomes.

Expected Valuations Consider a machine that flips out quarters one at a time. You and a friend come up to the machine and decide to split up the quarters. You take any quarter that comes out heads up, and your friend takes any quarter that comes out tails up. Since, on average, half of the time the quarters will come out heads and half the time they will come out tails, you expect to collect up to half of the quarters and your friend will take the other half. After the machine has flipped out ten quarters, you expect to have collected five of them and your friend will have collected the other five. After ten flips, you should have \$1.25 in quarters. This averages out to twelve and one-half cents per flip. So each time the machine spits out a quarter, it is worth twelve and one-half cents to you. Keep in mind that you never get twelve and one-half cents; you either get a quarter or nothing at all. So where does this value of twelve and one-half cents come from?

When the machine tosses a quarter, half of the time you will get a quarter and the other half of the time you will get nothing at all. The value you expect from each toss is one-half of a quarter and one-half of nothing at all. This calculation is called the expected value. It is

determined by multiplying the probability of an outcome by the value of that outcome and adding it to the probability of the next outcome multiplied by the value of the next outcome and so on until all possible outcomes are exhausted.

In general:

$$\begin{aligned} \text{Expected Value} &= \text{Probability Outcome 1} \times \text{Value Outcome 1} \\ &+ \text{Probability Outcome 2} \times \text{Value Outcome 2} \\ &+ \text{Probability Outcome 3} \times \text{Value} \dots \end{aligned}$$

Attaching a value to the outcome of the quarter-flipping machine:

$$\text{Expected Value} = (\frac{1}{2}) (25 \text{ cents}) + (\frac{1}{2}) (0 \text{ cents}) = 12.5 \text{ cents.}$$

Given this mechanism for attaching an expected valuation (EV) to a well-understood random process like the quarter-flipping machine, this can be applied to the siege game with random outcomes. Starting with the *Game of Siege* as illustrated in Figure 3.1, we can introduce outcome randomness by recognizing that a siege is not always successful. That is, the city does not automatically capitulate to the raiders if the city has chosen a course of action to close its gates, and the raiders do not automatically destroy the city guard if the city leaders have chosen to attack. In this new formulation of the siege game, the outcomes associated with the raiders choosing their course of action to siege are uncertain, as they are determined by the random process assigned to the nonstrategic player, Nature, as shown in Figure 3.12.

For the sake of this example, the outcome valuations accruing to the city and the raiders when Nature resolves the interaction of a siege to a success for the raiders is the same as posited in the original siege game in Figure 3.1. If the raiders choose their course of action to lay siege and Nature causes the interaction to fail, the raiders receive no loot but have exerted effort. In both cases, whether the siege is against a city that has chosen to close or chosen to attack, the raiders' failed siege achieves an outcome valuation of one. Additionally, the city, having prevailed over a raiders' siege, experiences an outcome valued at three

if its leaders chose their course of action to close the gates and a slightly lower outcome valued at two if they chose their course of action to attack. These outcome valuations are indicated in Figure 3.12.

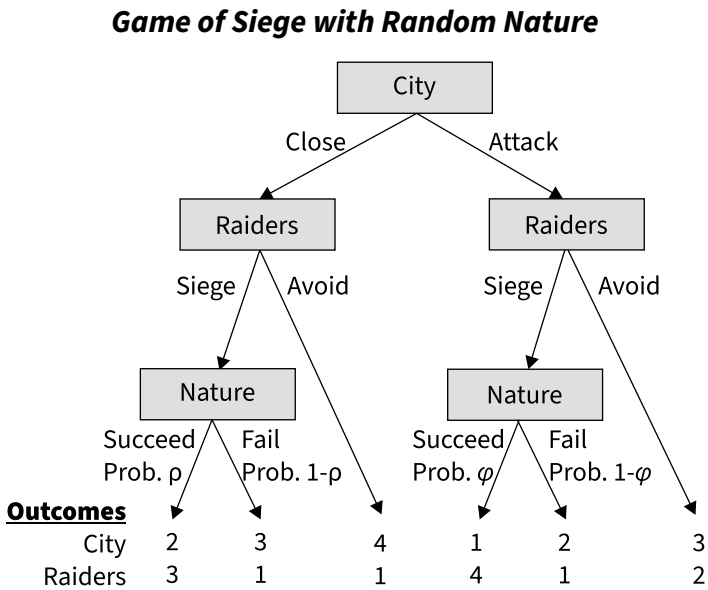


Figure 3.12: In this iteration of the siege game, the outcome of the raiders choosing their course of action to siege is uncertain. The siege, from the raiders’ perspective, can succeed with a probability (represented by p or ϕ) and fail with a probability ($[1-p]$ or $[1-\phi]$), where $0 \leq p$ or $\phi \leq 1$, depending upon the course-of-action choice by the city.

In addition to knowing the outcomes associated with the raiders choosing their course of action to siege, the two players need to know the probability distribution attributed to Nature across a successful siege and a siege failure. For the sake of this example, the probability of success in which the raiders siege a closed city is 50 percent, or $p = 0.5$, and the probability of success in which the raiders lay siege to an attacking city is 90 percent, or $\phi = 0.9$. This fully specified siege game with Nature acting in a random manner is shown in Figure 3.13.

Game of Siege with Random Nature

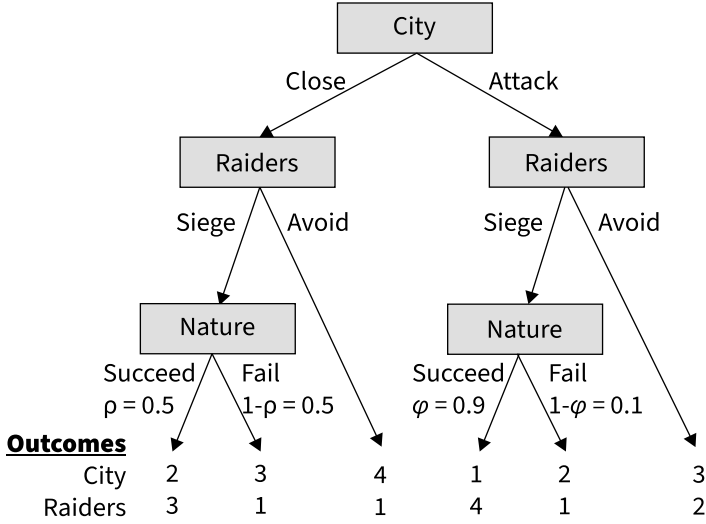


Figure 3.13: Specifying the probability distributions of the nonstrategic player Nature, where the probability of siege success is $p = 0.5$ and $\phi = 0.9$.

To perform an analysis of this sequential-play game, like the earlier version of this game, we must break it down into its first sub-games. But, unlike in the initial game, the siege game with Nature is involved in the furthest-down decision nodes. Nature is not a strategic player, so it cannot have best responses; it resolves to an outcome only in a probabilistic manner. In the discussion on expected valuation, we developed a technique determining the valuation of Nature’s probabilistic process. In Nature’s first decision node or left-hand node in Figure 3.13, the raiders’ siege will be successful with probability $p = 0.5$ and will fail with probability $(1-p) = 0.5$. The raiders’ expected outcome valuation (here represented by EV) for choosing their course of action to lay siege, given the fact that the city leaders have chosen to close their gates, is

$$EV_{\text{Raiders}}(\text{siege}) = ([\text{Probability Success}] \times [\text{Outcome to raiders of success}]) + ([\text{Probability Fail}] \times [\text{Outcome to raiders of Fail}]),$$

as in the following:

$$EV_{\text{Raiders}}(\text{siege}) = (0.5 \times 3) + (0.5 \times 1) = 2.0.$$

And the city's expected outcome valuation is

$$EV_{\text{City}}(\text{siege}) = (0.5 \times 2) + (0.5 \times 3) = 2.5.$$

Similar calculations to find the raiders' and the city's expected valuations in Nature's second or right-hand node yield

$$EV_{\text{Raiders}}(\text{siege}) = (0.9 \times 4) + (0.1 \times 1) = 3.7, \text{ and}$$

$$EV_{\text{City}}(\text{siege}) = (0.9 \times 1) + (0.1 \times 2) = 1.1.$$

The expected valuations for the city and the raiders are inserted in place of the nonstrategic player Nature's decision nodes, and the *Game of Siege* is simplified up one level to the raiders' decision nodes, as indicated in Figure 3.14. This provides a sequential-play game that is similar to the one discussed when introducing sub-game perfect equilibria, with two sub-games at the raiders' two decision nodes, as shown in Figure 3.14.

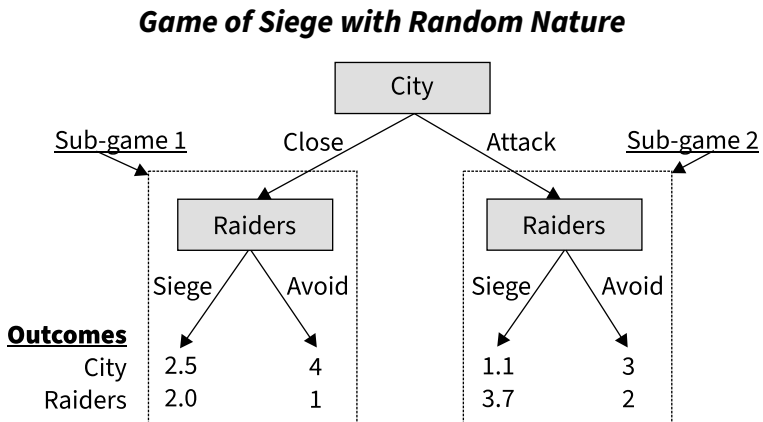


Figure 3.14: The random outcomes determined by Nature are resolved up, or replaced with Nature's expected outcome valuations, and the game is simplified up to the raiders' decision nodes. Once resolved up one level, the raiders' individual sub-games are apparent and indicated as Sub-game 1 and Sub-game 2.

To resolve the raiders’ two sub-games, the raiders will select the course of action that is the best response to the city’s course-of-action choice that led to that sub-game. So, in Sub-game 1, the raiders’ best response is to choose their course of action to lay siege. Similarly, the raiders’ best response in Sub-game 2 is to choose their course of action to lay siege as well. The raiders’ best responses in Sub-game 1 and Sub-game 2 are highlighted in Figure 3.15. Conjointly, the raiders’ irrational course-of-action choices are pruned, or grayed out, as also indicated in Figure 3.15.

Moving up the extensive form of the *Game of Siege with Nature* to the city’s decision node, the city leaders know the raiders’ best-response course-of-action choices to each of the city’s courses of action. The city’s best response is to close its gates. This best-response course-of-action choice close is highlighted in Figure 3.15, with the irrational course-of-choice attack grayed out.

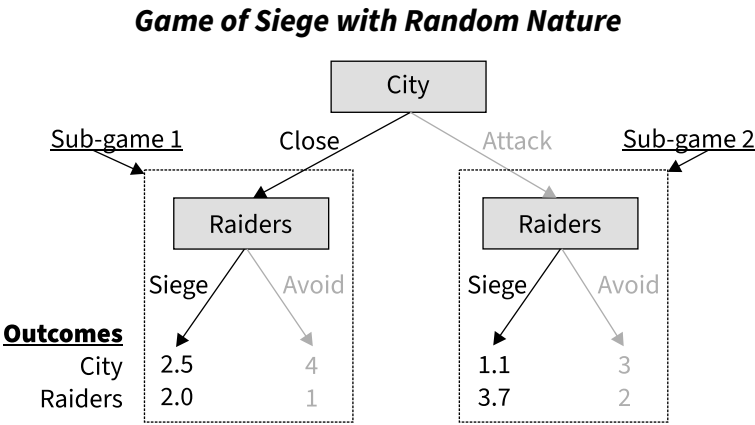


Figure 3.15: The raiders’ course-of-action choice, in either Sub-game 1 or Sub-game 2, will be a best response to the city’s course-of-action choice that resulted in the raiders being in Sub-game 1 or Sub-game 2. In both instances, the raiders’ course-of-action choice is to siege. Once the raiders’ best responses for their sub-games are established, the city’s best response is established, whereby the city chooses its course of action to close.

The sub-game perfect Nash equilibrium—in which all courses of action made down the decision-node chain of the extensive form of the game are mutual best responses—is the highlighted path in Figure 3.15. This equilibrium finds that the city closes and the raiders choose to siege in response to the closure or to siege if the city attacks, yielding an expected outcome valuation of 2.5 to the city and an expected outcome valuation of two to the raiders.

The Equilibrium to *Operation Overlord/Fortress Europe* Having built out the fundamentals of finding sub-game perfect Nash equilibria to sequential-play games by using backward induction, this process can be applied to the game introduced in Chapter 2, *Operation Overlord/Fortress Europe*, a game with a nonstrategic player, Nature, that acts randomly. In the previous sections of this chapter, the expected value operator is shown as one way to attach valence to a set of well-defined random outcomes. To calculate expected outcome valuations for the game *Operation Overlord/Fortress Europe* as described in Figure 2.6 and presented again in Figure 3.16, outcome valuations need to be provided for all the possible branches of the extensive form of the game. Since there are only three possible outcomes for this simple version of the game, there is an easy value scheme. Germany would prefer repelling the invasion to allowing the Allies a foothold, but it would prefer a foothold to the Allies breaking through. The Allies' preferences would be exactly the other way around. Thus, a simple outcome ranking would assign a value of three to the most preferred outcome (for Germany, repelling the invasion would take a value of three, and the Allies' breakout would be ranked three), working down to a ranking of one for the players' least preferred outcome. Figure 3.16 represents the game with the outcome valuations in place of the actual outcomes. Germany's valuation is displayed first, followed by the Allied valuation. Given the outcome valuations and the probability of realizing each of the outcomes for both the Allies and Germany, it is possible to calculate the expected value to each player at each of Nature's nodes. For example, in the very

Operation Overlord/Fortress Europe

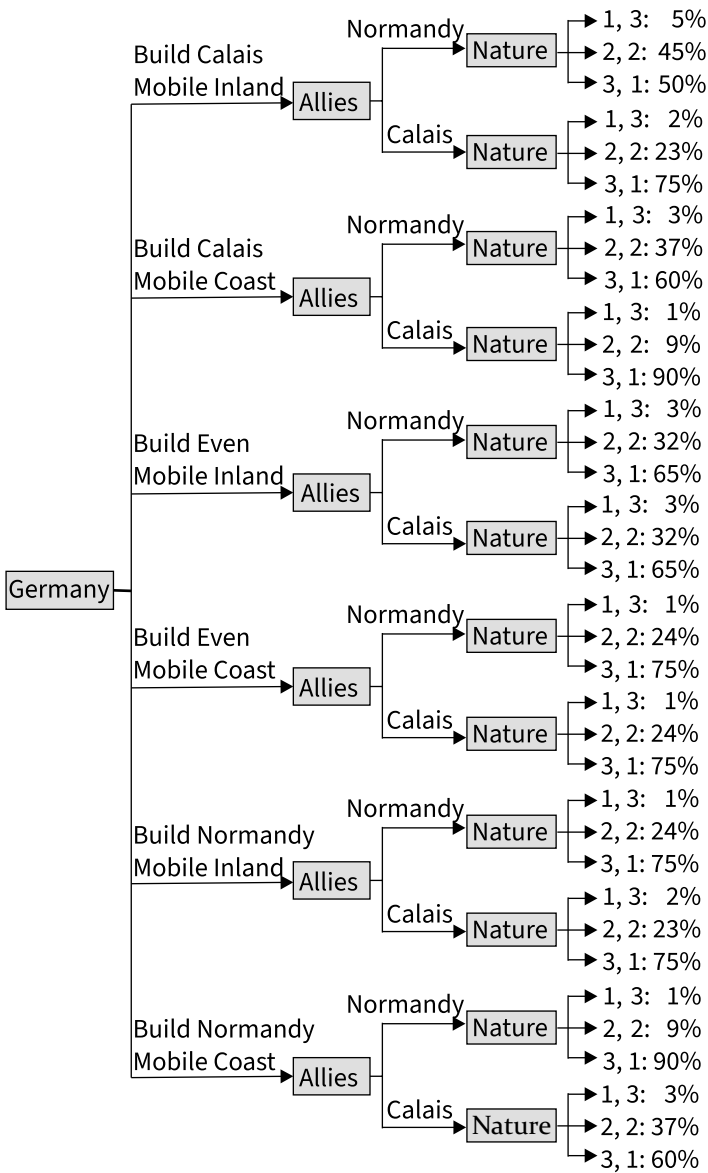


Figure 3.16

top Nature node of Figure 3.16, Germany has outcome valuations of one, two, and three, with probability of 0.05, 0.45, and 0.50, respectively. Using the expected value calculation, Germany has an expected outcome value of

$$EV_{\text{Germany}}(\text{Normandy}) = (0.05)(1) + (0.45)(2) + (0.50)(3) = 2.45.$$

The Allies have outcome valuations of three, two, and one, with probability of 0.05, 0.45, and 0.50, respectively. The Allies have an expected outcome value of

$$EV_{\text{Allies}}(\text{Normandy}) = (0.05)(3) + (0.45)(2) + (0.50)(1) = 1.55.$$

Using these expected valuations associated with the random choice of Nature, the game can be resolved up a level as in Figure 3.17. With the new extensive-form representation of the game *Operation Overlord/Fortress Europe* in Figure 3.18, the technique of backward induction can be applied to determine the sub-game perfect equilibriums to the game. Starting at the outcomes, look up one level to the Allied decision nodes to find the sub-games. There are six sub-games. In each of these six sub-games, the Allies are deciding between invading Europe at Calais or Normandy. The Allies will choose the invasion site that will give them the highest-expected outcome valuation.

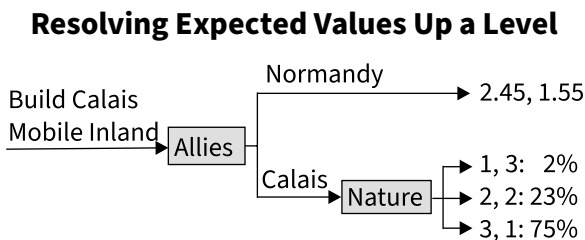


Figure 3.17: The expected value of the outcomes determined by a random player are calculated; then the expected values are used as a proxy for that random player's decision node. This resolves the extensive form of the game up a level.

Operation Overlord/Fortress Europe

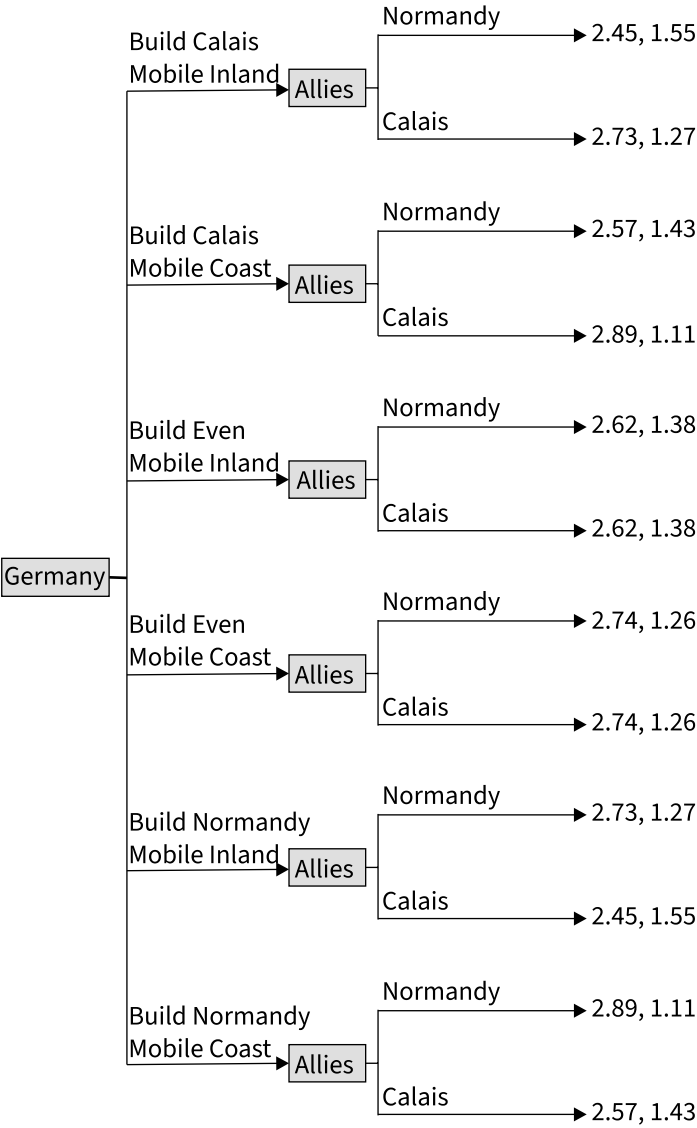


Figure 3.18

In the very top sub-game in Figure 3.18, the Allies can choose between invading at Normandy, with an expected outcome value of 1.55, and invading at Calais, with an expected outcome value of 1.27. The Allies will choose the Normandy invasion with its expected outcome value of 1.55 in this sub-game. Notice that in the middle two sub-games, the expected outcome value of both a Normandy and Calais invasion are the same to the Allies. Thus, in these two sub-games, the Allies are indifferent to their choices of invasion sites. In this representation of the game *Operation Overlord/Fortress Europe*, the Allies are indifferent to the choice of invasion site if the Germans choose to build their coastal defenses evenly.

Continuing as previously done in the *Game of Siege with Nature*, the first-level sub-games are solved. Courses of action that the Allies would *not* choose are eliminated in all six of the Allies' sub-games. This yields the game represented in Figure 3.19. Here the Allies' less-preferred course-of-action choices have been eliminated, leaving only their best responses to each of Germany's course-of-action choices.

The expected outcome values that remain are then moved up a level in the game tree. Germany now chooses a course of action given the Allies' best response to Germany's course-of-action choice. Figure 3.20 represents the top sub-game with Germany's decision node. In this formulation of the game, Germany would choose to build up fixed defenses evenly at Calais and Normandy and would have its mobile armor and infantry forward at the coast. The expected outcome valuation to Germany would be 2.74.

Building *Operation Overlord/Fortress Europe* back up using the information from resolving the sub-games and the expected valuations yields the game tree represented in Figure 3.21. There are two sub-game perfect equilibriums in this game. They are Germany chooses to build evenly with its mobile forces forward to the coast, while the Allies choose to either invade at Normandy or at Calais. Both equilibrium courses of action yield expected outcome valuations of 2.74 to Germany and 1.26 to the Allies.

Operation Overlord/Fortress Europe

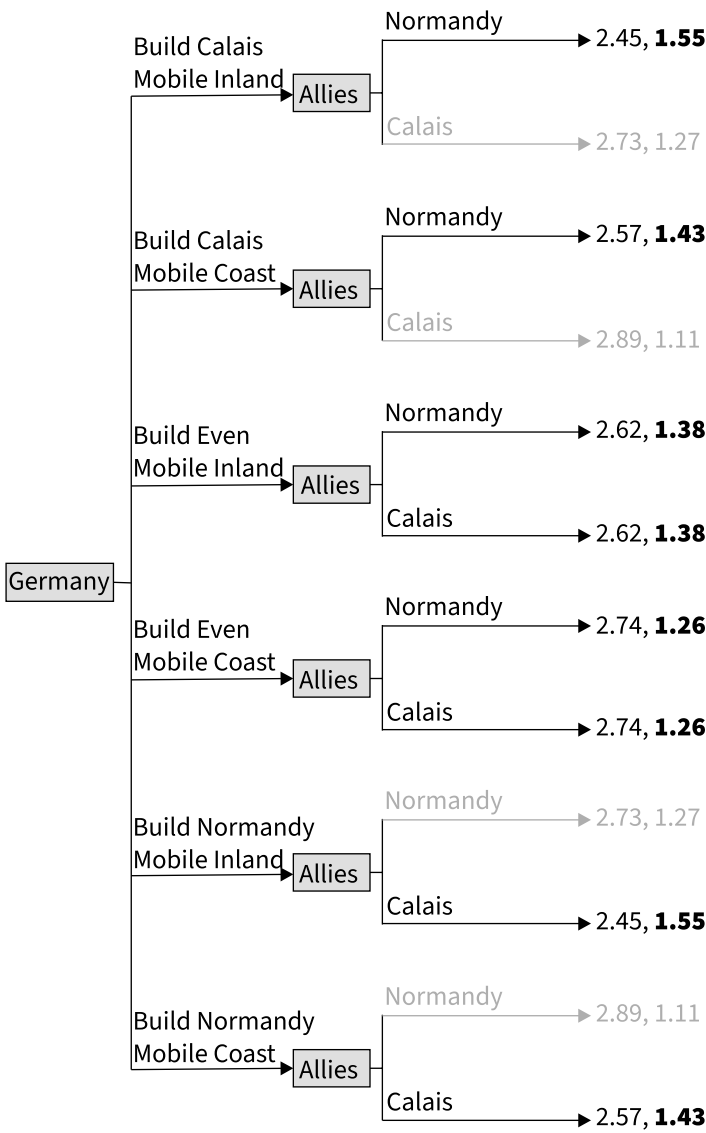


Figure 3.19

Operation Overlord/Fortress Europe

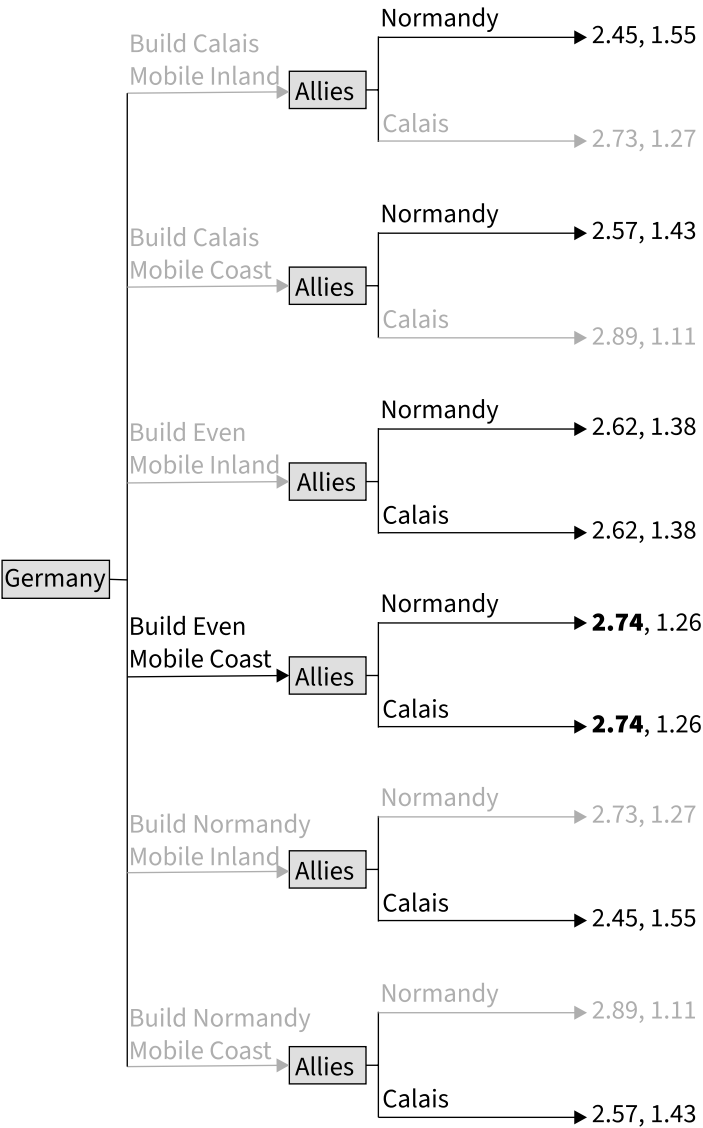


Figure 3.20

Operation Overlord/Fortress Europe

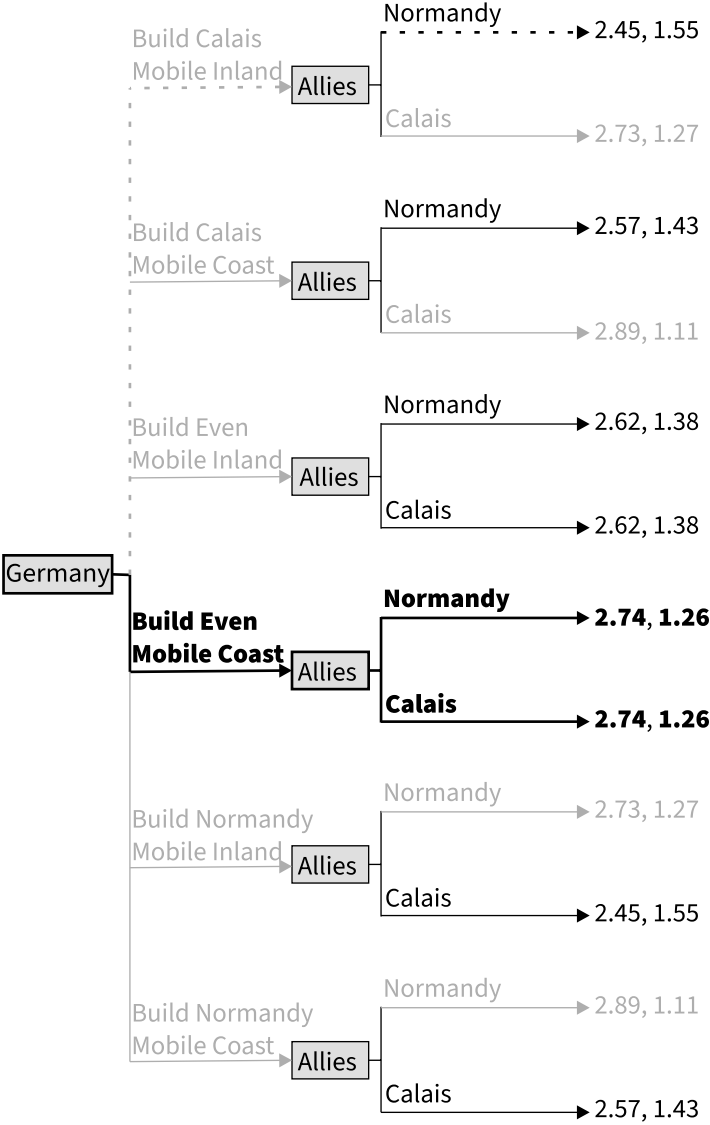


Figure 3.21

Formally, the full description of the contingent courses of action of the sub-game perfect Nash equilibria are:

1. Germany chooses its course of action to build its defenses evenly with its mobile defenses at the coast, while the Allies choose their contingent courses of action whereby: (1) the Allies invade at Normandy, if Germany has built its fixed coastal defenses evenly and deployed its mobile defenses at the coast; (2) the Allies invade at Normandy if Germany chooses to build at Calais with its mobile defenses inland; (3) the Allies invade at Normandy if Germany builds its defenses at Calais with its mobile defenses at the coast; (4) the Allies invade at Normandy or at Calais if Germany builds its defenses evenly with its mobile defenses inland; (5) the Allies invade at Calais if Germany builds its defenses at Normandy with its mobile defenses inland; and (6) the Allies invade at Calais if Germany builds its defenses at Normandy with its mobile defenses at the coast.
2. Germany chooses its course of action to build its defenses evenly with its mobile defenses at the coast, while the Allies choose their contingent courses of action whereby: (1) the Allies invade at Calais, if Germany has built its fixed defenses evenly with its mobile defenses deployed to the coast; (2) the Allies invade at Normandy if Germany chooses to build at Calais with its mobile defenses inland; (3) the Allies invade at Normandy if Germany builds its defenses at Calais with its mobile defenses at the coast; (4) the Allies invade at Normandy or at Calais if Germany builds its defenses evenly with its mobile defenses inland; (5) the Allies invade at Calais if Germany builds its defenses at Normandy with its mobile defenses inland; and (6) the Allies invade at Calais if Germany builds its defenses at Normandy with its mobile defenses at the coast.

As an interesting historical note, looking back on records of World War II, this is the course of action advocated by the German commander Erwin Rommel but overridden in favor of building up Calais and holding the German mobile forces inland, as indicated with the dashed lines in Figure 3.21.⁷ That this simple model of *Operation Overlord/Fortress Europe* supports a proposed German strategy is both coincidental and indicative of the potential game theory has as a tool in the arsenal of a strategic decision-maker.

If Rommel had had access to game theory as an analytical tool, he may have been better able to demonstrate the strengths of his plan to Field Marshal Gerd von Rundstedt, his immediate superior. If von Rundstedt had access to game-theoretic modeling, he could have explained to Rommel the circumstances that made his plan superior.

It is foolish to assume that Field Marshal von Rundstedt was strategically inept and was not competent to choose the German course of action to defend Europe. Given the game presented that supported Field Marshal Rommel's strategic vision, a reasonable question to ask is what game was Field Marshal von Rundstedt seeing that made his course-of-action choice reasonable? Game theory allows us to answer that question by building an alternative interpretation of the strategic-decision environment that supports Field Marshal von Rundstedt's course-of-action choice. In doing so, we will see how game theory can help planners solidify their underlying presuppositions as to the world in which they reside.

CHAPTER IV

BELIEFS AND SCENARIOS

If I know our troops can attack, but do not know the enemy cannot be attacked, it is only halfway to victory.

If I know the enemy can be attacked, but do not realize our troops cannot attack, it is only halfway to victory.

Thus it is said if you know them and know yourself, your victory will not be imperiled.

SUN TZU, *ART OF WAR*

In the game *Operation Overlord/Fortress Europe* as described in Chapter III, it appears that Germany chooses a course of action to build its defenses at Calais and to keep its mobile defenses inland. That action leads to an outcome that is of a lower outcome valuation when compared with an alternate course-of-action choice available. An initial assessment of Germany's strategic decision-making could be that the German planners are behaving irrationally. Though there were many instances in which Adolf Hitler did override his military planners, this situation involves a fundamental disagreement between two German theaters in Europe. Allied strategic planners relying on facing an irrational opponent when developing a strategy and making course-of-action choices are choosing to operate in an incomprehensible environment where planning serves no purpose.

An alternative way to consider this potentially irrational outcome is to ask, knowing that the game as presented in Chapter III represents the core understanding that Field Marshal Rommel has of our game *Operation Overlord/Fortress Europe*, how does Field Marshal von Rundstedt see the same situation? Assuming that von Rundstedt and Rommel both saw the exact same structure of course-of-action choices and sequences as presented in Figure 3.16 and reproduced here as Figure 4.1, the difference between the two could be their individual beliefs about the probabilities associated with the probability distributions across each of Nature's "decision" nodes.

A game with differing belief structures across probability distributions is said to contain multiple scenarios. In Rommel's scenario, his beliefs regarding the probability distributions over outcomes at Nature's decision nodes were informed by his earlier command experience in North Africa fighting against the British. In that prior engagement, the British were able to bring airpower against Germany's

tank forces. British airpower was extremely capable at interfering with German tank movement.¹ Given the Allies' air supremacy in Europe at the time of our game *Operation Overlord/Fortress Europe*, the probability distributions across Nature's decision nodes in the game assume that tanks held in reserve further inland would not be easily moved to an Allied invasion site and would reduce the probability of Germany repelling an Allied invasion. They also increase the probability that an Allied invasion would be successful. Therefore, it was with Rommel's understanding that the probabilities in Figure 4.1 were developed.

Field Marshal von Rundstedt's previous combat experience was early in the war, on Germany's eastern front with the Soviet Union. At the time, the Russians had no significant airpower. Von Rundstedt could move tanks quite freely across the eastern front without airpower interference. With his Russian-front combat background, von Rundstedt's beliefs about the mobility of the tanks held in reserve in Europe would result in an increase in the probability of a German counter-attack outside of the range of Allied battleship fire and a decrease in the probability that an Allied invasion would be successful.²

The Rommel scenario of *Operation Overlord/Fortress Europe* was built by applying the standard three-to-one personnel ratio needed for a 50 percent chance of attacker victory to Rommel's beliefs regarding the likelihood that the fighting forces held in reserve inland could reach the Allied invasion point in a timely manner. The von Rundstedt scenario uses the same personnel ratio but also his beliefs regarding the likelihood that the inland reserves will reach an Allied invasion point in a timely manner. We cannot know his actual belief structure and thus his probability distributions across Nature's decision nodes, but we can imply that, if von Rundstedt behaves rationally, Germany's expected outcome valuation from his course-of-action choice would exceed Germany's expected outcome valuation in the equilibrium presented in Figure 3.21, reproduced here as Figure 4.2.

Rommel Scenario Operation Overlord/ Fortress Europe

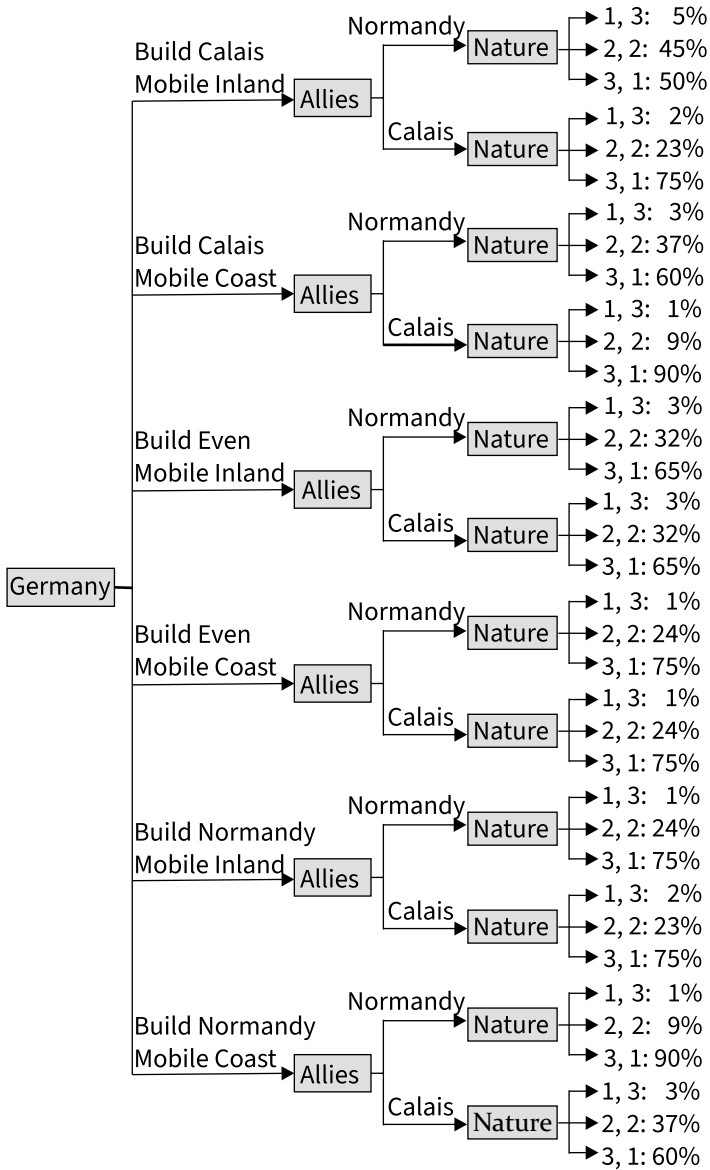


Figure 4.1

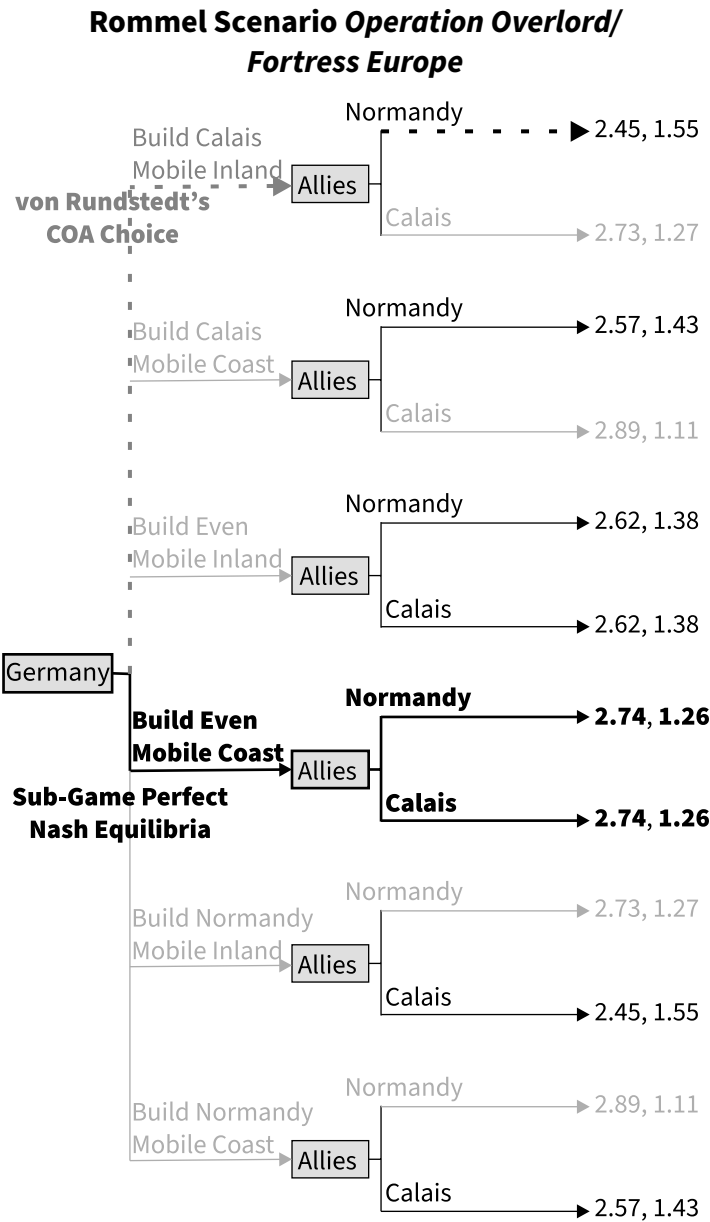


Figure 4.2

For the sake of the von Rundstedt scenario, we are going to make the simplifying assumption that the only difference between Rommel and von Rundstedt's beliefs is their understanding of the efficacy of mobile forces held inland in reserve. Thus, their beliefs about Nature's probability distributions for branches of the game in which mobile forces are positioned forward at the coast are the same in both of their scenarios and will have the same expected outcome valuations as reported in Figure 4.2. Additionally, von Rundstedt's beliefs about Nature's probability distributions are reported in Figure 4.3.

That said, von Rundstedt's beliefs regarding Nature's probability distributions must be consistent with the fact that he overrode Rommel's recommendation of having Germany's mobile forces move forward. Thus, a rational von Rundstedt would perceive a scenario of the game in which Germany's expected outcome valuations are higher along the mobile inland decision paths than the decision paths where mobile forces are at the coast, given the best response of the Allies. We must also note that von Rundstedt's actual course-of-action choice during the war to build Calais's mobile forces inland must have an expected outcome valuation of at least 2.74. Von Rundstedt's choice must at least match the German equilibrium outcome valuation of the Rommel scenario of *Operation Overlord/Fortress Europe*.

Given von Rundstedt's beliefs about the probability distribution across Nature's outcomes in Figure 4.3, the expected outcome valuations for the game *von Rundstedt Scenario Operation Overlord/Fortress Europe* can be calculated. In the top branch of that game, in which Germany chooses its course of action to build Calais's mobile forces inland and the Allies choose their course of action to attack at Normandy, Germany has an expected outcome valuation of $EV_{\text{Germany}}(\text{Normandy}) = (0.05)(1) + (0.15)(2) + (0.80)(3) = 2.75$. The Allies, however, have an expected outcome valuation of $EV_{\text{Allies}}(\text{Normandy}) = (0.05)(3) + (0.15)(2) + (0.80)(1) = 1.25$. All the expected outcome valuations displayed for *von Rundstedt Scenario Operation Overlord/Fortress Europe* are presented in Figure 4.4.

***von Rundstedt Scenario Operation Overlord/
Fortress Europe***

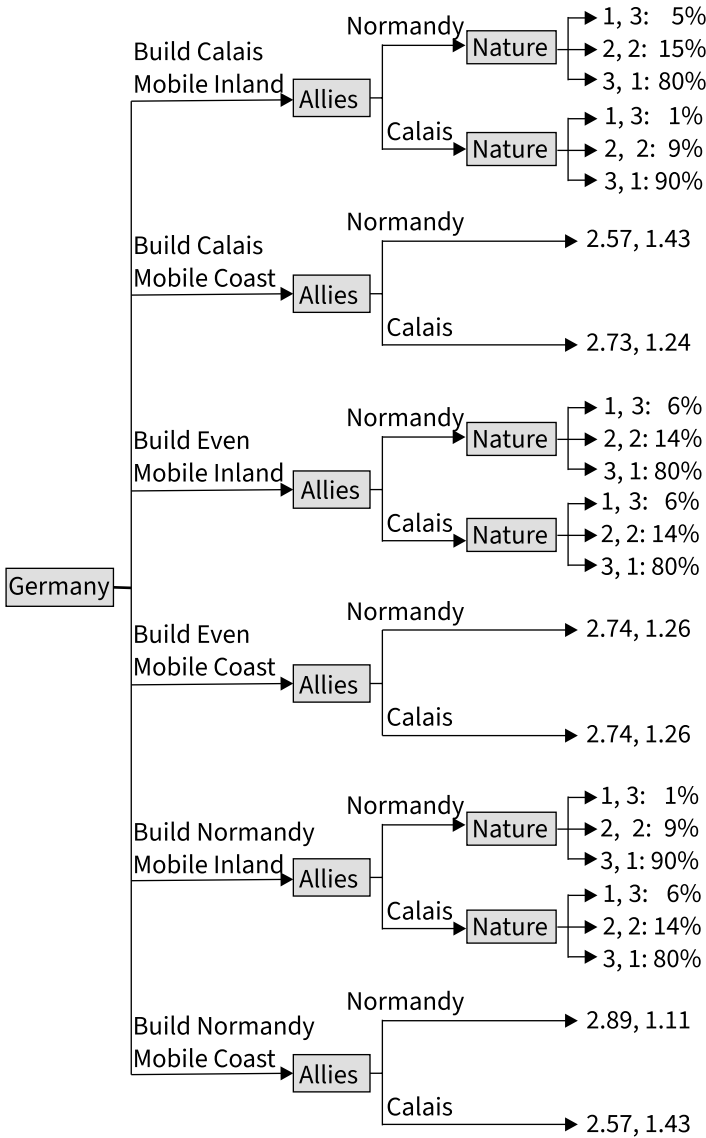


Figure 4.3

***von Rundstedt Scenario Operation Overlord/
Fortress Europe***

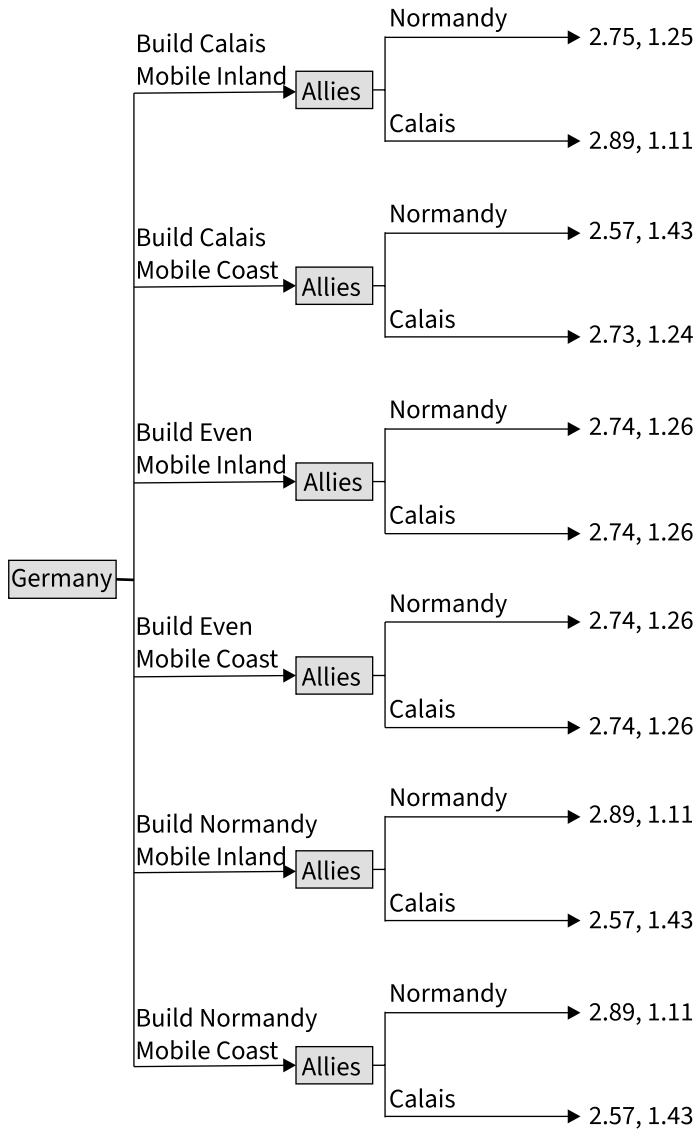


Figure 4.4

Once Nature's random play is reduced to expected outcome valuations, the six deepest level of sub-games can be evaluated. Each of these sub-games is at the Allies' decision node to choose between their courses of action at Normandy and/or Calais as illustrated in Figure 4.5. The Allies, if they find themselves in Sub-game 1 in Figure 4.5, know that Germany has chosen its course of action to build fixed forces at Calais with mobile forces held inland and face the course-of-action choice to invade Normandy with an expected outcome valuation of 1.25, or the course-of-action choice to invade Calais with an expected outcome valuation of 1.11. Being rational, the Allies, finding themselves in Sub-game 1, choose their course of action to invade Normandy. The Allies' preferred course-of-action choice to invade Normandy is indicated in Figure 4.5 by the graying out of their less attractive course of action to invade Calais.

In each of the six sub-games extending from each of the German course-of-action branches, the Allies will select their best response to the German course-of-action choice. The graying out of the Allies' less preferable course-of-action choices, in each of their six sub-games, leaves the Allies' best response as depicted in Figure 4.6.

Given the Allies' best-response course-of-action choices, Germany's sub-game of the *von Rundstedt Scenario of Operation Overlord/Fortress Europe* is formalized and can be analyzed.

With each of the Allies' six sub-games resolved, this game moves up one level to Germany's sub-game, starting at the German course-of-action choice node. Given the Allies' best responses at each of their downstream decision nodes, in Germany's sub-game, Germany will make its course-of-action choice that gives it the highest-expected outcome valuation, and its other course-of-action choices will be grayed out, as depicted in Figure 4.6. Germany will choose its course of action to build fixed forces at Calais and keep mobile forces inland, yielding an expected outcome valuation of 2.75. Germany will choose this course of action because it is its best response, given the Allies' downstream best responses.

***von Rundstedt Scenario Operation Overlord/
Fortress Europe***

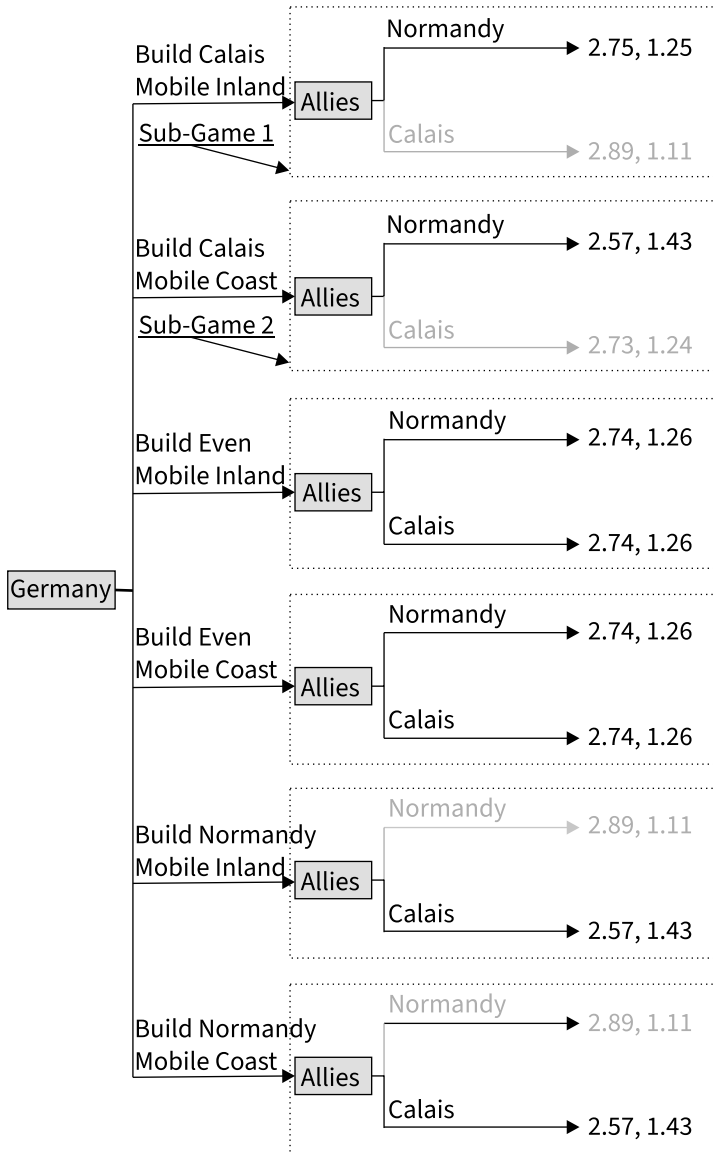


Figure 4.5

**von Rundstedt Scenario Operation Overlord/
Fortress Europe**

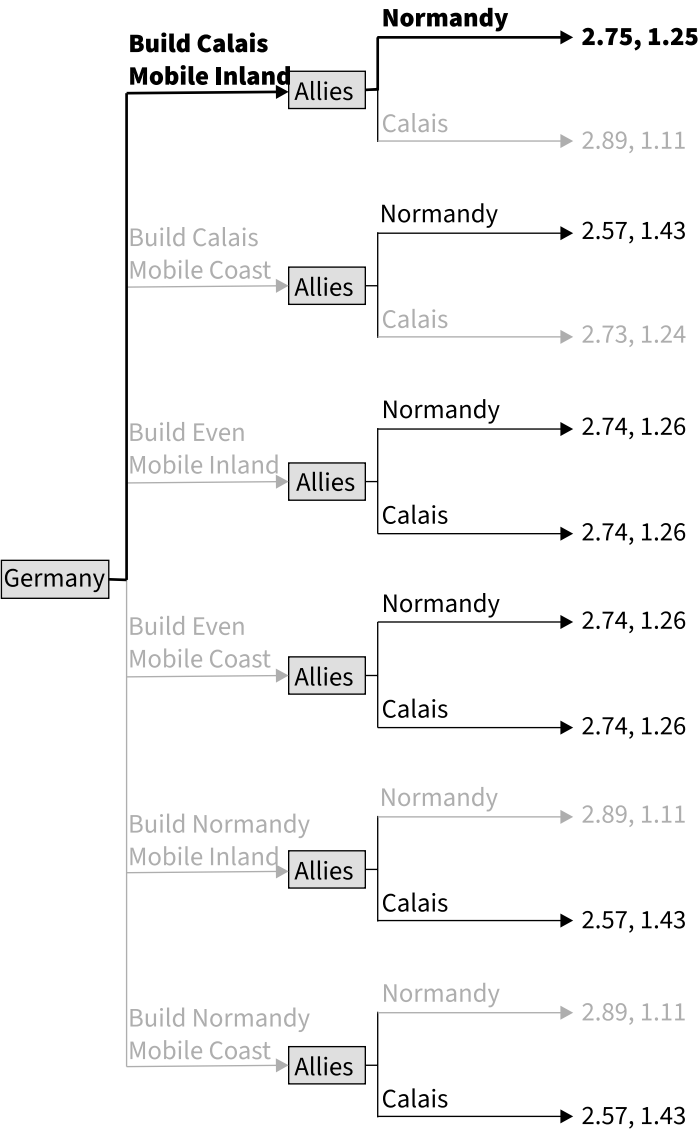


Figure 4.6

The bold pathway highlighted in Figure 4.6 presents the sub-game perfect Nash equilibrium to the *von Rundstedt Scenario Operation Overlord/Fortress Europe*. Given von Rundstedt's beliefs regarding the probability distributions across the random player Nature's outcomes, the game is expected to resolve with Germany choosing its course of action to build fixed forces at Calais while holding its mobile forces inland and the Allies choosing to invade at Normandy, with Germany achieving an expected outcome valuation of 2.75 and the Allies achieving an expected outcome valuation of 1.25. The formal equilibrium contingent course-of-action choice set is Germany chooses its course of action to build Calais mobile inland while the Allies choose their contingent course of action: Normandy, if Germany chooses to build Calais mobile inland; Normandy, if Germany chooses to build Calais mobile coast; Normandy or Calais, if Germany chooses to build even mobile inland; Normandy or Calais, if Germany chooses to build even mobile coast; Calais, if Germany chooses to build Normandy mobile inland; and Calais if Germany chooses to build Normandy mobile coast.

The players' contingent strategies described above meet the requirements of a sub-game perfect Nash equilibrium because the players are choosing their best-response courses of action in each of the game's sub-games.

With von Rundstedt's understanding of the mobility of German tanks under Allied air superiority affecting his beliefs about the probability distribution of the random player Nature's outcomes, *von Rundstedt Scenario Operation Overlord/Fortress Europe* can have a sub-game perfect Nash equilibrium consistent with the course-of-action choice that played out in the actual invasion at Normandy during World War II. In Chapter III, using Rommel's understanding of the mobility of German tanks under Allied air superiority, a different sub-game perfect Nash equilibrium of the game emerged. In the strategic planning stage of any endeavor, there are competing scenarios of the strategic environment. Germany faced at least two scenarios for Fortress Europe. Using game theory to frame the discussion over the appropriate course-of-action choice for Germany would require that Rommel

and von Rundstedt articulate their assumptions and focus on the actual differences between their strategic visions.

Rommel could lay out his experience with British air superiority in North Africa and its ability to obstruct tank movement, while von Rundstedt could lay out his eastern-front experience against the Soviet Union and the relative ease of tank movement. The two could then present and discuss their separate scenarios of the game, with a clear understanding of the assumptions underlying their divergent constructions. They could then restrict their analysis to resolving their opposing understandings. Perhaps von Rundstedt acquiesces to Rommel's North Africa experiences. Perhaps Rommel concedes to von Rundstedt's eastern-front experiences. Either way, the ability to build a theoretical game construct focuses on the strategic analysis of both the cause and the relevance of what causes conflicting interpretations.

Of course, it would be foolish to assume that the oversimplified game model presented here to the reader is a sufficient representation of the strategic-choice arena faced by the Allies and the Germans. For example, a common ploy used in military engagements is to feign attack at one location, causing the defensive force to adjust or commit its forces, only to attack in earnest elsewhere. The Germans would have been foolish not to anticipate the possibility of misdirection as a part of an Allied attack.

The next chapter is a case study that applies the processes developed in these early chapters to the Battle at Gettysburg, fought in 1863 during the American Civil War. We will use the history of General Robert E. Lee's incursion into Pennsylvania and the subsequent battle at Gettysburg to build a game-theoretic model of the strategic decision-making of General Lee and General George Meade. We will use game theory to find the sub-game perfect Nash equilibrium and to tease out insights into how the battle played out. The game-theoretic structure will be oversimplified because the main point of this case study is to build a game consistent with the concepts currently covered. The secondary goal is to provide an example of how to craft a game consistent with the information available to the strategist.

CHAPTER V

CASE 1: THE BATTLE FOR GETTYSBURG

In antiquity those that excelled in warfare made themselves unconquerable in order to await [the moment when] the enemy could be conquered.

*Being unconquerable lies with yourself;
being conquerable lies with the enemy.*

Thus, one who excels in warfare is able to make himself unconquerable but cannot necessarily cause the enemy to be conquerable.

Thus, it is said a strategy for conquering the enemy can be known but yet not possible to implement.

SUN TZU, *ART OF WAR*

During the Battle for Gettysburg in the U.S. Civil War, the 72,000-person-strong Army of Northern Virginia (ANV) fought the 100,000-person-strong Union Army of the Potomac, and between 46,000 and 51,000 Americans died, were wounded, or went missing.¹ However, the epic clash between the United States of America and the Confederate States of America (CSA) at Gettysburg, Pennsylvania, from July 1 to July 3, 1863, failed to achieve the decisive victory that each side desperately wanted. The prospect that both General Robert E. Lee, commanding the Army of Northern Virginia, and General George Meade, commanding the Union Army of the Potomac, would commit so many soldiers and so much equipment to a battle without significant strategic planning is difficult to imagine.

The game-theoretic tools developed in the preceding chapters will be applied to the Battle of Gettysburg. By applying a game-theoretic construct to this battle, the military will gain a better understanding of why leaders need to think strategically in all tactical engagements, regardless of past experiences, in order to protect themselves from making errors similar to the ones Lee made.

Despite his successful experiences in employing defensive tactics, Lee pursued an inferior course of action against Meade at Gettysburg. What made his decision to attack even more questionable was his relative lack of success employing offensive tactics during the campaigns and battles in the year preceding Gettysburg. Had his desire to engage in a decisive battle clouded his decision-making abilities? Did the information available to him support an attack? Were there other options available to Lee? These questions need to be addressed while maintaining an awareness that the processes of war include not only the tactical-level fight, but also the strategic and operational levels.

A close look at the battlefield suggests that Lee had the information available to him to choose a different course of action, as well as enough information to weigh those alternative courses of action against his decision to attack. Had he decided to defend in place instead of attack during the initial stages of Gettysburg, the outcome may have been drastically different.

The analysis presented here shows that the Battle of Gettysburg should not have been the high-water mark of the Civil War. However, the reality is that after the third day of fighting at Gettysburg, Lee had lost up to 28,000 or about one-third of his fighting force,² the initiative, and the morale of his army in pursuit of a course of action that he should not have pursued. Those losses ensured the South would never again pose a serious threat to Northern soil. Beyond the impact Gettysburg had on the outcome of the Civil War, Lee's choices serve as an example of why strategic decision-making and game theory should be used for tactical application.

I. Background on the Battle at Gettysburg

Lee's decision to attack at Gettysburg was based on a series of external events that began when Lee was appointed to the position of commander of the Army of Northern Virginia. During the year that preceded Gettysburg, Lee had fought five significant campaigns/battles. The chapter will begin with a synopsis of these five campaigns/battles to provide an understanding of the events that may have contributed to how and why Lee chose his strategy and tactics.

i. History of Lee's Campaigns

Lee assumed command of the Army of Northern Virginia on May 31, 1863. Selected by Jefferson Davis, the Confederate president, to succeed the wounded General Joseph Johnston, Lee's war record could be characterized as undistinguished to that point.³ However, he would

rewrite his war record beginning with the Seven Days Campaign, when he faced off against General George McClellan and the Army of the Potomac. This was to be the first of five significant battles or campaigns Lee would fight prior to facing Meade in Gettysburg. In each of the five, although confronting an enemy with superior numbers, Lee gained the tactical victory. While these victories transformed his “undistinguished” war record to “dazzling,” in the words of the historian Stephen W. Sears,⁴ Lee’s army could not continue to win such Pyrrhic victories because the Union Army’s numbers were too great for the South to survive a war of attrition. The first such victory was the Seven Days Campaign.

In the Seven Days Campaign, which began on June 25 and lasted through July 1, 1862, McClellan had moved the Union forces to within striking distance of Richmond, Virginia, the capital of the Confederate States of America.

Lee wasted no time before going on the offensive to preclude a siege of his capital. On June 26th, Lee attacked the flank of the defending Union Army. Lee’s forces suffered 1,500 casualties while the North suffered only 400 in this initial attack, but Lee had forced the Union to pull back to a more defensible position. A day later, Lee pressed the attack and broke through the Union lines. Still faced with the threat of a capable Union Army near Richmond, Lee continued the attack on June 29th and 30th, despite the fact that McClellan had occupied the high ground, Malvern Hill, in defense. In each of the first five major skirmishes of the Seven Days Campaign, Lee continuously pummeled McClellan’s forces and lost more men than McClellan. By the sixth skirmish, despite attacking uphill against McClellan in the defense, Lee was able to take Malvern Hill and force McClellan to leave the Richmond area and move toward Harrison Landing. Lee had shown his tenacity by repelling the larger Army of the Potomac from the southern capital but at the cost of a quarter more casualties than those suffered by the North. In the offense, Lee was tactically victorious, but McClellan maintained the strategic advantage by leaving the battle with a large and capable force. For the North, this ended

the Peninsula Campaign. For the South, Richmond was no longer threatened.

Continuing to move to the North, Lee's forces would take the field against a larger Union force in the Second Battle of Bull Run, also known as Second Manassas. The significant skirmish took place on August 29th and 30th, when General Thomas Jonathan "Stonewall" Jackson would successfully defend against the Union's forces. Lee began the battle with 55,000 forces, or two-thirds the 70,000 forces that General John Pope used to attack.⁵ By the end of the battle, Pope's attacking forces lost almost twice as many men as did Lee in the defense. This victory is the first of three successive defensive victories for Lee.

By the middle of September, Lee's forces had pushed as far as Sharpsburg, Maryland, and had occupied a defensive position along Antietam Creek. McClellan then moved his forces to the opposite side of the creek and began an attack on Lee's defenses.

The result was the single bloodiest day in American history, with both sides losing a total of more than 23,000 casualties. While Lee is credited with a tactical victory for having fought to a draw against a much larger force brought by McClellan, this victory again threatened the strategic capability of Lee's army. Unhappy with McClellan's failure, President Abraham Lincoln subsequently appointed General Ambrose E. Burnside as commander of the Army of the Potomac. In mid-November 1862, General Burnside sent one corps toward Fredericksburg.

In response to Burnside's movements, Lee moved his forces toward Fredericksburg and occupied a defensive position on the high ground overlooking the town. On December 11th, Burnside prepared to attack Lee by placing pontoon bridges across the Rappahannock River. The next day the Union Army crossed the river. Burnside committed his men to a string of frontal attacks against Lee's defenses for three days. By December 15th, Burnside had suffered 12,500 casualties, more than two times the 6,000 casualties suffered by Lee's forces. Burnside moved back across the Rappahannock in defeat.⁶ Lee had

The Battle at Antietam

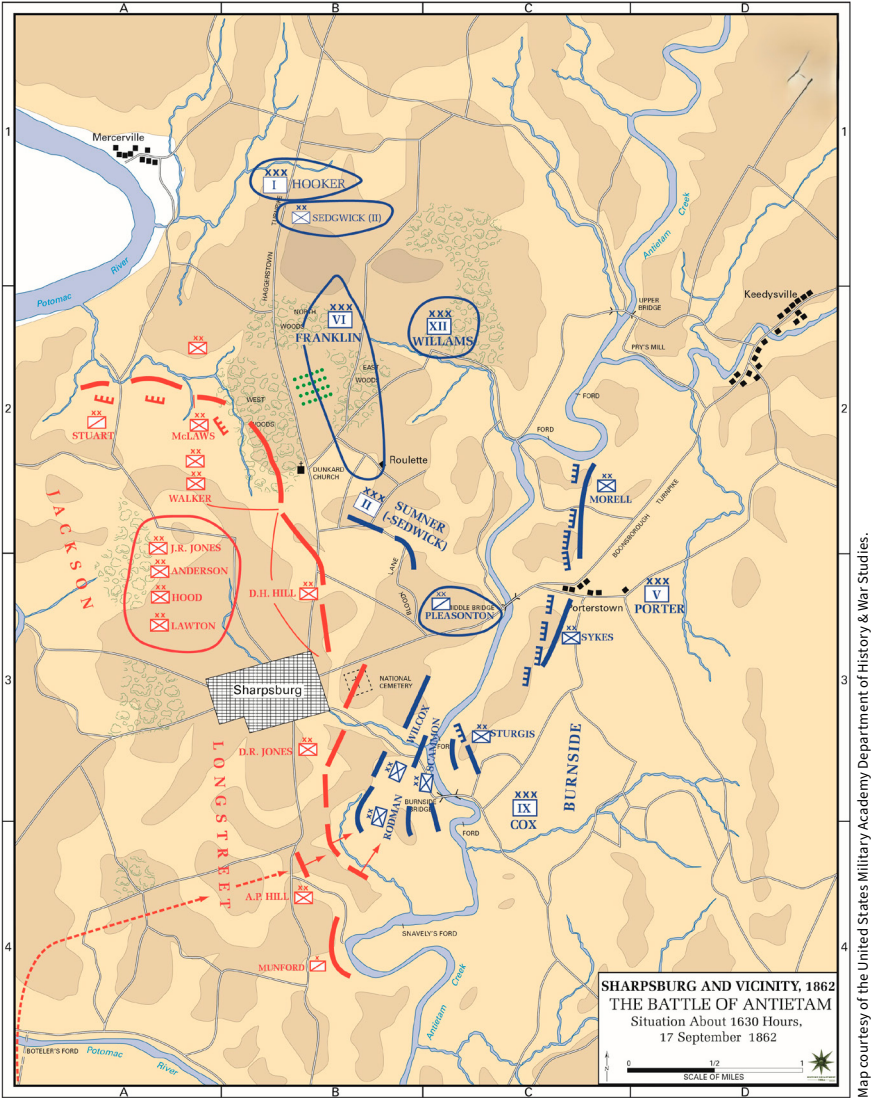


Image 5.1

The Battle of Fredericksburg

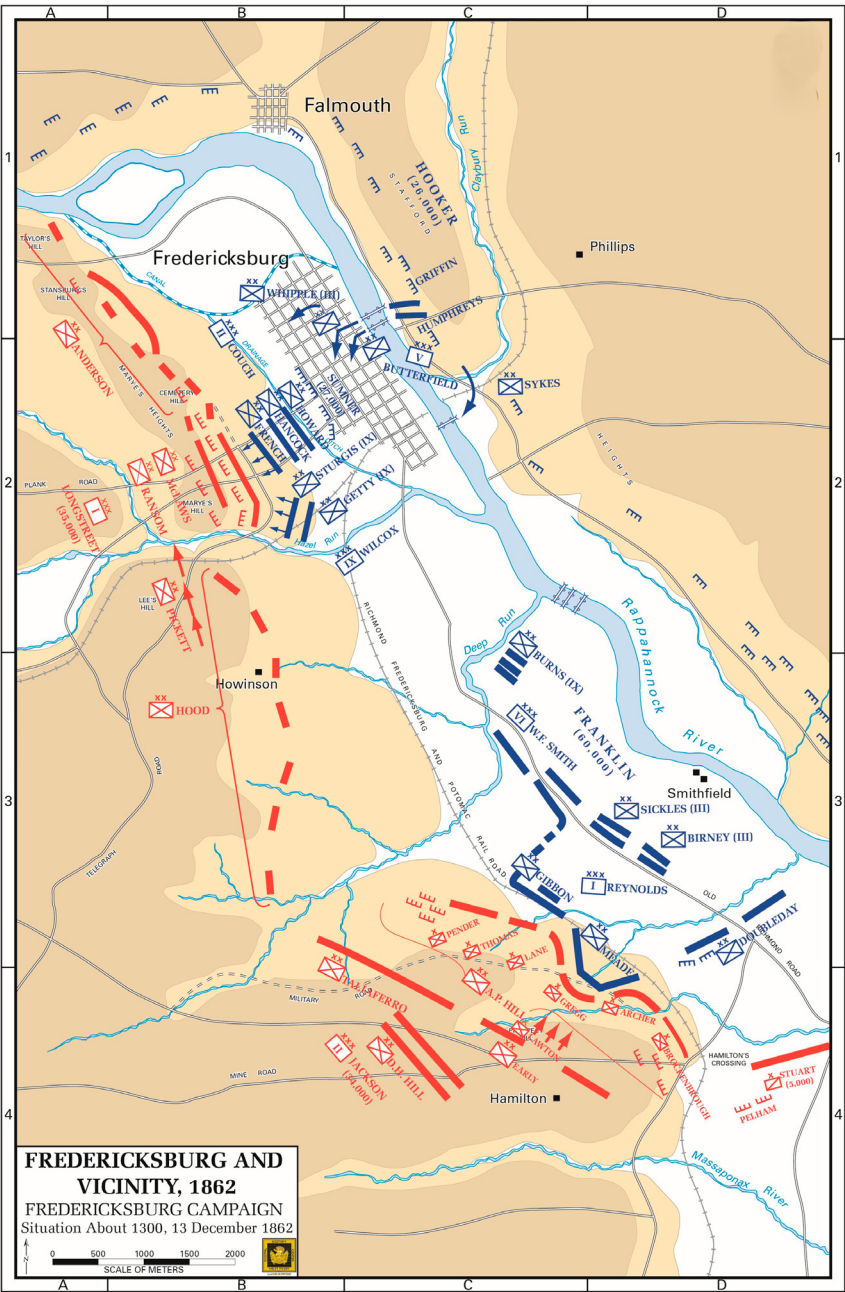


Image 5.2

defeated an army much larger than his by occupying the high ground and preparing a strong defense.

President Lincoln then replaced Burnside with General Joseph Hooker to command the Army of the Potomac. Hooker moved his forces to the west along the north side of the Rappahannock in an attempt to cross the river and draw Lee into a fight where Lee did not occupy such an advantageous position. In response, Lee left some forces in Fredericksburg and set out for a fight in the vicinity of Chancellorsville, Virginia. Both generals were keenly aware of the numbers that either side could bring to the battle. With about half as many men as Hooker, Lee made some assumptions about the enemy that made the disparity in numbers acceptable to him. First Lee assumed that Hooker would be no different from the other Northern generals he had already faced. Lee's second assumption was that although he had smaller numbers, he could build defenses in depth so that he could fall back into these defenses as the Union forces advanced. These defenses would negate the benefit Hooker had of having the greater number of troops. Lee's third assumption was that Hooker was about to lose between 30,000 and 40,000 men because their enlistment was due to run out in a few months, and Lee did not think they could be pressed into battle. Hooker was confident that he would defeat Lee's much smaller force by forcing him into terrain more favorable to Hooker's own forces. In the open, Hooker felt his forces could decisively defeat Lee's.

As Union corps began moving to push Lee's forces out of the wilderness area, the dense forests and heavy undergrowth limited the Union's ability to engage Lee's elements head-on. Instead, the Union forces were unable to force Lee's troops out of the easily defensible terrain. Over the next three days of fighting, Hooker attempted to push Lee out of his defenses only to be repelled each time.⁷ By May 3rd, Hooker was spread over a wide front and his center was preparing to defend. Lee—given the opportunity to attack or retreat with victory in hand—chose a daring attack. He would send more than half his men, under Jackson's command, to find the right flank of the opposing forces, while he, with only 16,000 men, would attack the center of the 70,000-strong Union army.

The Battle at Chancellorsville May 1

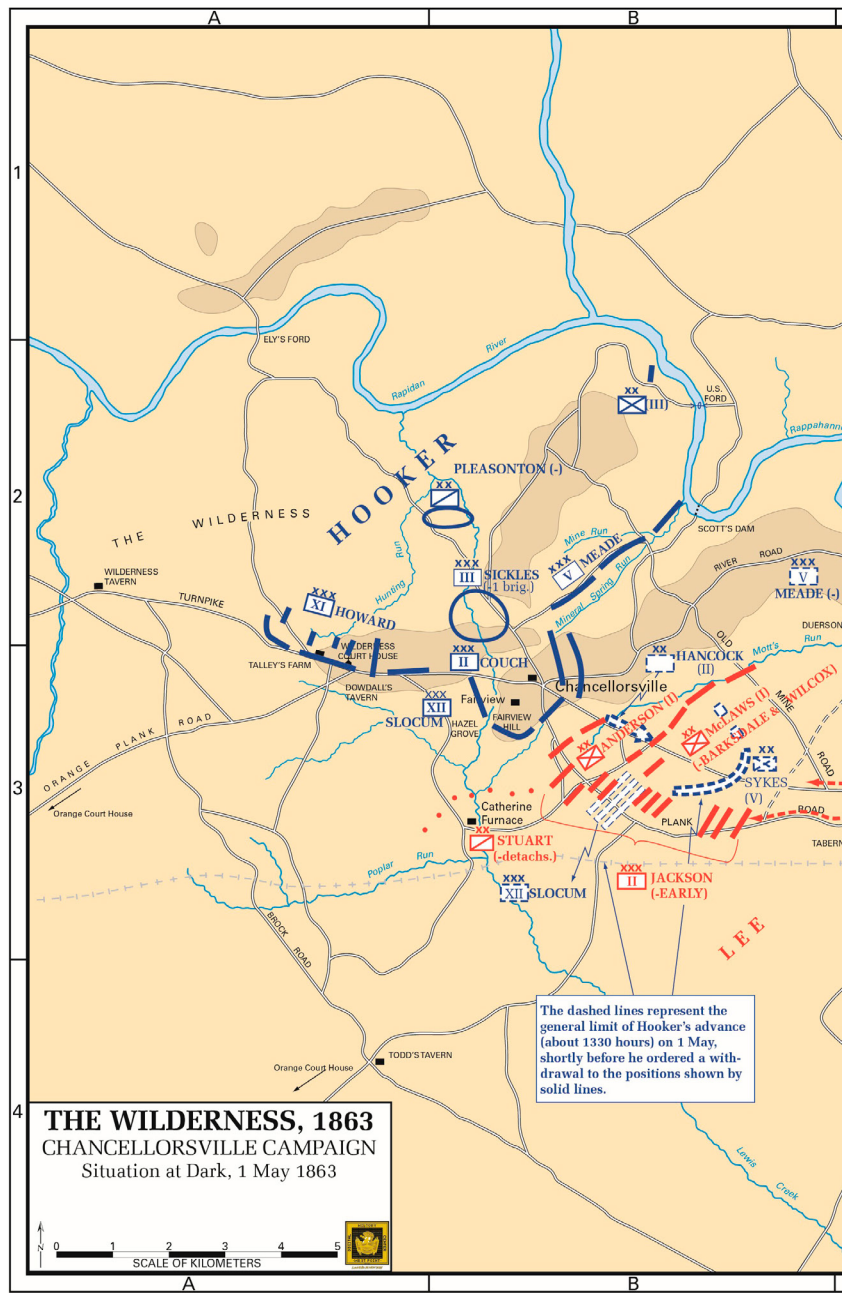
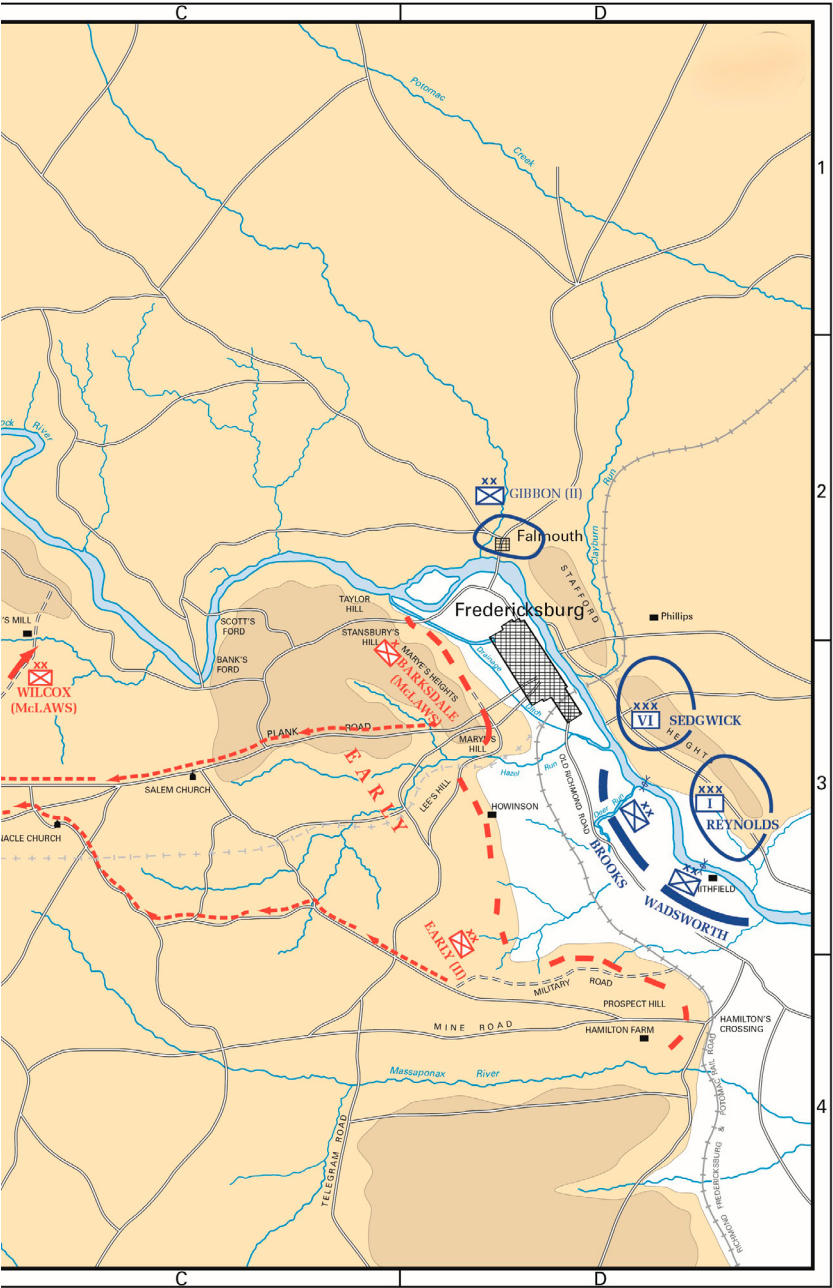


Image 5.3



Map courtesy of the United States Military Academy Department of History & War Studies.

The Battle at Chancellorsville May 4

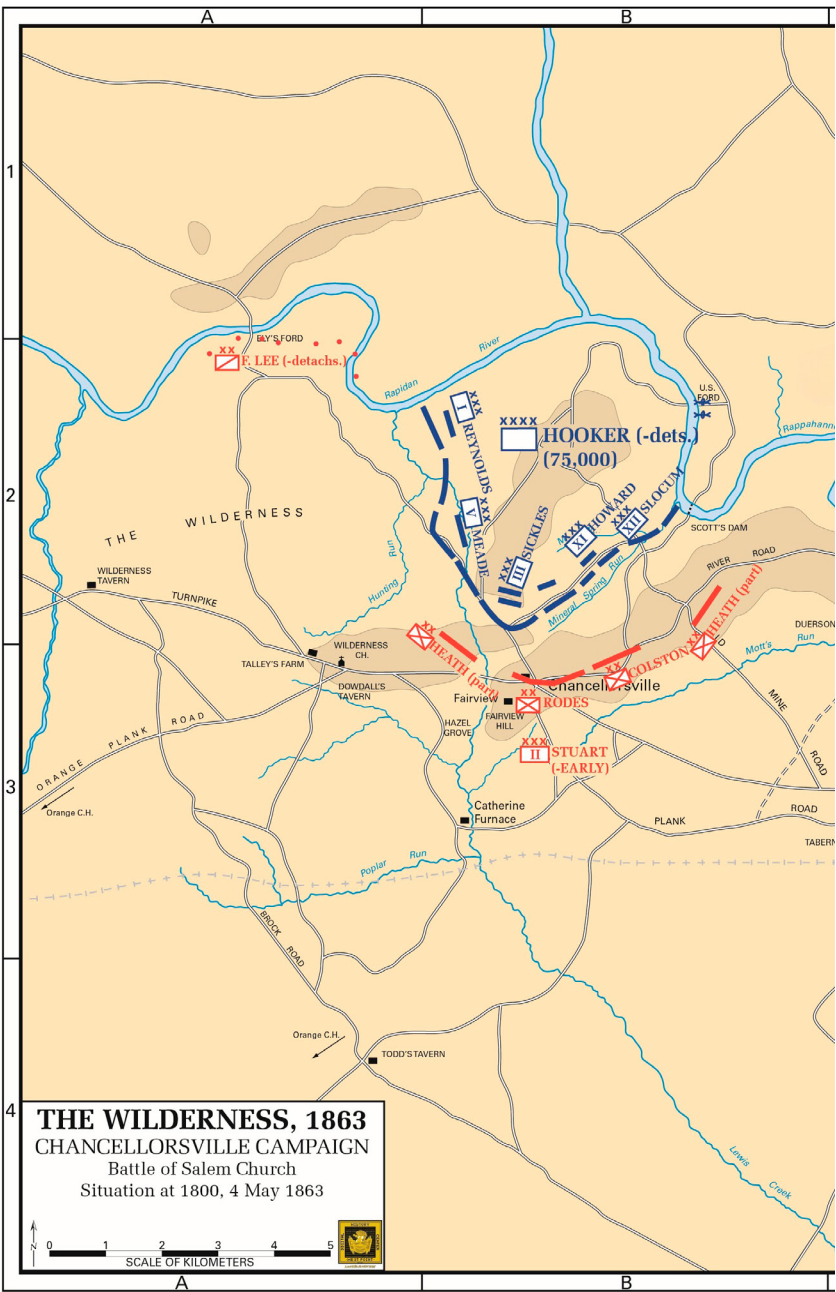
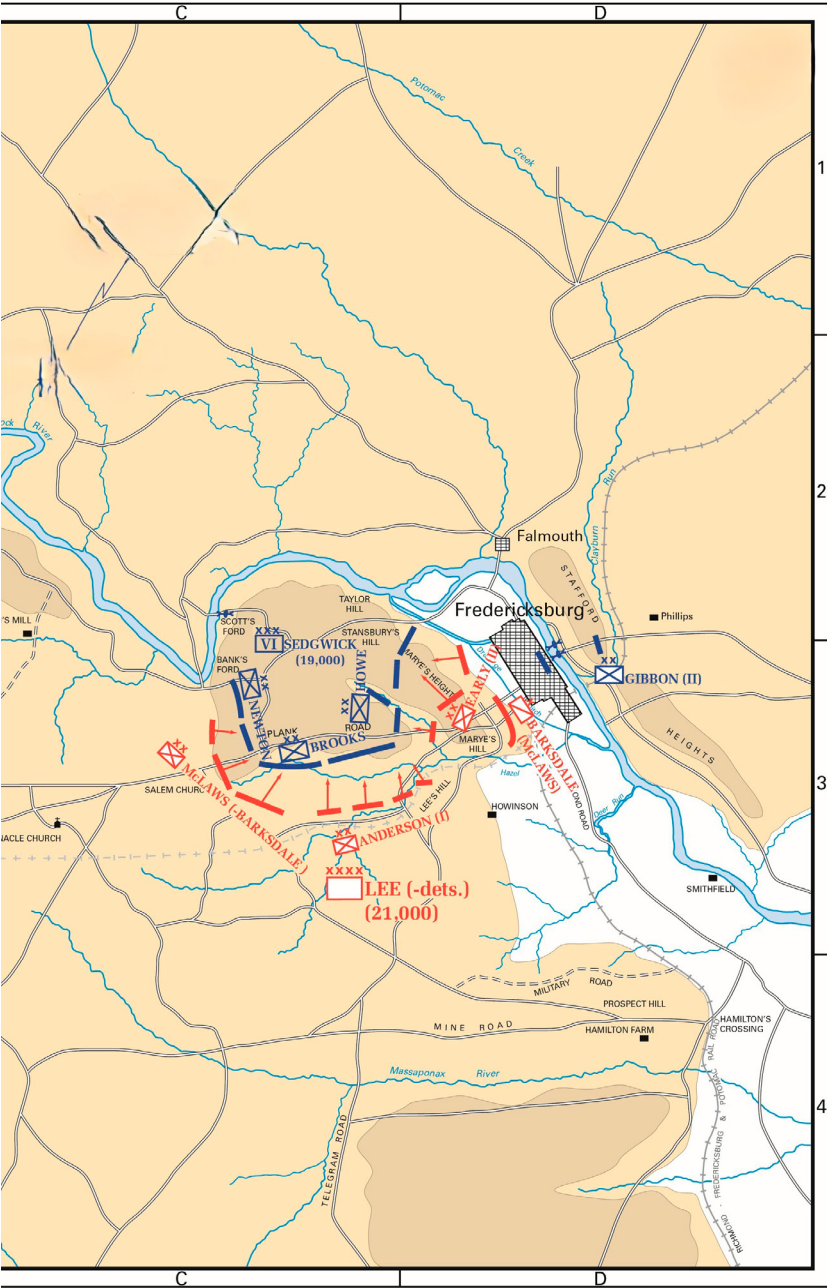


Image 5.4



Map courtesy of the United States Military Academy Department of History & War Studies.

The terrain that kept the Union forces from engaging Lee on a wide front now protected Lee as he focused his 16,000 men on a very small front of the Union defenses and allowed Jackson's element to move with surprise and attack the flank. Repeated assaults forced Hooker to order a retreat by the night of May 5th. By then, the battlefield had claimed almost 30,000 casualties.

Sears explains that some of Hooker's officers, "knowing that Lee was still greatly outnumbered, urged Hooker to launch a counter-attack."⁸ For this and other failures, Hooker would be relieved of command and replaced with General George Meade. Lee, having won resounding victories in Fredericksburg and Chancellorsville, "was heard to say that Chancellorsville depressed him even more than Fredericksburg had: 'Our loss was severe, and again we had gained not an inch of ground and the enemy could not be pursued.' What he wanted in [the] future," Sears goes on to write, "was battle on his terms, on ground of his choosing, with no barriers to a final outcome."⁹ According to Confederate General James Longstreet, they had experienced a series of "fruitless victories; . . . even victories such as these were consuming us, and would eventually destroy us."¹⁰

ii. Gettysburg

Lee understood that the successes he had achieved on the field of battle could not be sustained, especially since he was sure the Union would continue to grow in manpower and equipment strength.¹¹ He was convinced that a decisive strike in the heartland of the North would bring victory and that the successes of the recent battles had tipped the scales in his favor. Comparatively, Lee's army was at the "height of its power and confidence . . . and the North was at its weakest point."¹² Fully aware of his relative weaknesses in numbers, Lee felt he had to be the strategic aggressor. However, that strategic aggression did not imply offensive operations. As interpreted by Sears, "Indeed, in the best execution of the idea, it would mean just the opposite—marching and maneuvering so aggressively on campaign that Lee might accept

battle or not, as he chose, with his opponent forced to give battle—to attack—at a time and in a place of Lee’s choosing.”¹³ Or, in the words of Sun Tzu:

As for constricted configurations, if we occupy them first we must fully deploy throughout them in order to await the enemy. If the enemy occupies them first and fully deploys in them, do not follow them in. If they do not fully deploy in them, then follow them in.¹⁴

Longstreet captured the intent by stating “the ruling ideas of the campaign may be briefly stated [as] thus . . . under no circumstances were we to give battle but exhaust our skill in trying to force the enemy to do so in a position of our own choosing.”¹⁵ Lee set his plan in action by beginning movement on June 3rd. By June 16th, most of the Army of Northern Virginia was in Maryland. By the end of that month, Lee’s forces were in Pennsylvania, with the state capital at Harrisburg the objective.

On June 27, 1863, President Lincoln relieved Hooker and replaced him with Meade. When he was told of his appointment, Meade is said to have replied, “Well, I’ve been tried and condemned without a hearing, and I suppose I shall have to go to the execution.”¹⁶ As the newly appointed commander of the Army of the Potomac, Meade was not afforded the luxury of time and experience to plan a grand strategy against an old foe. He had to move quickly to pursue Lee, while keeping his forces between Lee’s forces and Washington. Meade’s campaign plan was to destroy the Army of Northern Virginia. He would attempt to do that by massing his whole army against a portion of Lee’s forces.¹⁷

Meade had commanded a corps in the fighting in Chancellorsville and was familiar with Lee’s capabilities as a commander. Lee, with his contemptuous perceptions of the previous Northern commanders, was quoted as telling his staff, “Meade will make no mistake in my front . . . and if I make one, he will make haste to take advantage of it.”¹⁸

iii. The Historic Battle

For the Confederate forces, the movement north through Maryland and into Pennsylvania en route to Harrisburg was remarkably unimpeded. However, after the initial movements into the southern part of Pennsylvania, intelligence reports from a Confederate spy alerted Lee and Longstreet, Lee's most competent and trusted subordinate commander, of the Union's movement to the area around Gettysburg, some six hours from the Confederate position. Lee, envisioning the positive impact that a large-scale victory would have on the war efforts, began to consolidate the Army of Northern Virginia. His orders to his subordinates were to avoid contact with the Union if possible.¹⁹ He planned on massing his entire army against a portion of Meade's forces.

However, on June 30, 1863, Confederate troops, composed mainly of General Henry Heth's corps, spotted what appeared to be Union forces. The Confederates decided to attack the position the following day, when they discovered they were fighting Major General John Buford and two cavalry brigades.

The Confederate attack proved initially successful, but the infantry soon reinforced the Union position led by Major General John Reynolds and the First and Eleventh Corps.

Despite the reinforcements, the Confederacy eventually won the skirmish and felt the momentum shift in their favor. Meade, from his headquarters in Taneytown, about nine miles south of Gettysburg, received word of the skirmishes in Gettysburg. Believing Gettysburg to be Lee's objective, Meade committed to pursuing battle there.²⁰

Lee—although angry that Heth's corps had failed to disengage either because Heth was unwilling or unable to do so—believed that the first day's victory meant he had the initiative and needed to take action to retain the initiative.²¹ Unable to receive intelligence from his cavalry, Lee sought one of Longstreet's spies for information.²² The spy informed Lee of the massive movement of Northern forces toward the South's current location. He also told Lee of a forward element of cavalry troops that were standing alone without infantry and of the larger

element that was approximately a day behind the forward element. Further, the spy informed Lee that the Northern commander was Meade. Finally, the spy expressed his opinion that the lowered Northern morale and the expiration or near expiration of many Northern soldiers' enlistment time would benefit the South.²³

With the information received from the spy and from the first day's skirmish in mind, Lee calculated that the enemy comprised four corps of infantry with three more unaccounted for. Lee assumed that Meade would move his other three corps to Gettysburg, and that by the morning of July 2nd, the Union commander would have had eighteen hours' notice of the first day's battle. He reasoned that if Meade desired a decisive battle, he would have his other three corps moving before nightfall on July 1st. "If Lee was to attack with any advantage—his entire army against some fraction of Meade's—he could not afford to wait for [cavalry commander J. E. B.] Stuart or anyone else. 'A battle had, therefore, become in a measure unavoidable,' Lee would explain in his report, 'and the success already gained gave hope of a favorable issue.'"²⁴

Although Meade spent the afternoon of July 1st sending orders for his army to prepare to fight in Gettysburg, he was perplexed that Heth's corps was not preceded by cavalry. This led him to believe the North had run into the Confederate forces by accident and had been caught off guard or that the Confederate element was not large enough to warrant cavalry. To confuse matters more, the Confederate forces had overwhelmed Major General Buford's two cavalry brigades and the reinforcing infantry corps led by Major General Reynolds. What he gained from this skirmish was that there were two Confederate corps at Gettysburg. The third Confederate corps, commanded by Longstreet, was unaccounted for and gave him hope that Longstreet would not join the battle by Day 2. Nevertheless, Meade was committed to the fight even if Longstreet were to arrive.²⁵ Meade, as indicated by Sears, "hoped the next day to mass the whole of the Army of the Potomac at Gettysburg to attack some fraction of the enemy's army."²⁶ He assumed he would face two Confederate corps with his seven corps.²⁷

Gettysburg July 1

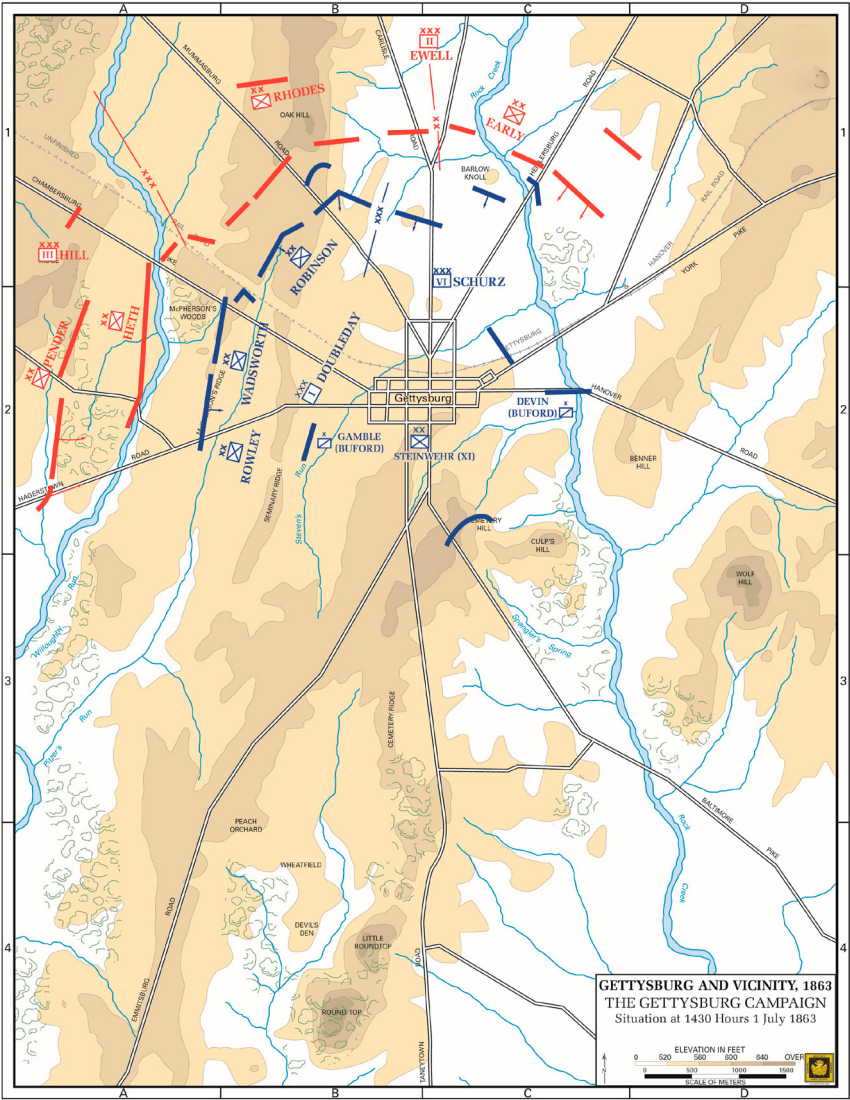


Image 5.5

By the end of the first day of fighting, the Union forces had been pushed south of Gettysburg and the Confederate forces had captured the town. However, the Union forces were able to occupy Cemetery Hill, a key piece of high ground overlooking the town. The Union prepared defenses on Cemetery Hill and nearby Cemetery Ridge as more reinforcements filled their ranks that night. By noon of Day 2 at Gettysburg, there were six corps from the Army of the Potomac occupying Cemetery Ridge. They were facing all three corps from Lee's Army of Northern Virginia occupying Seminary Ridge, which was about one mile to the west and roughly parallel to the Union position.²⁸

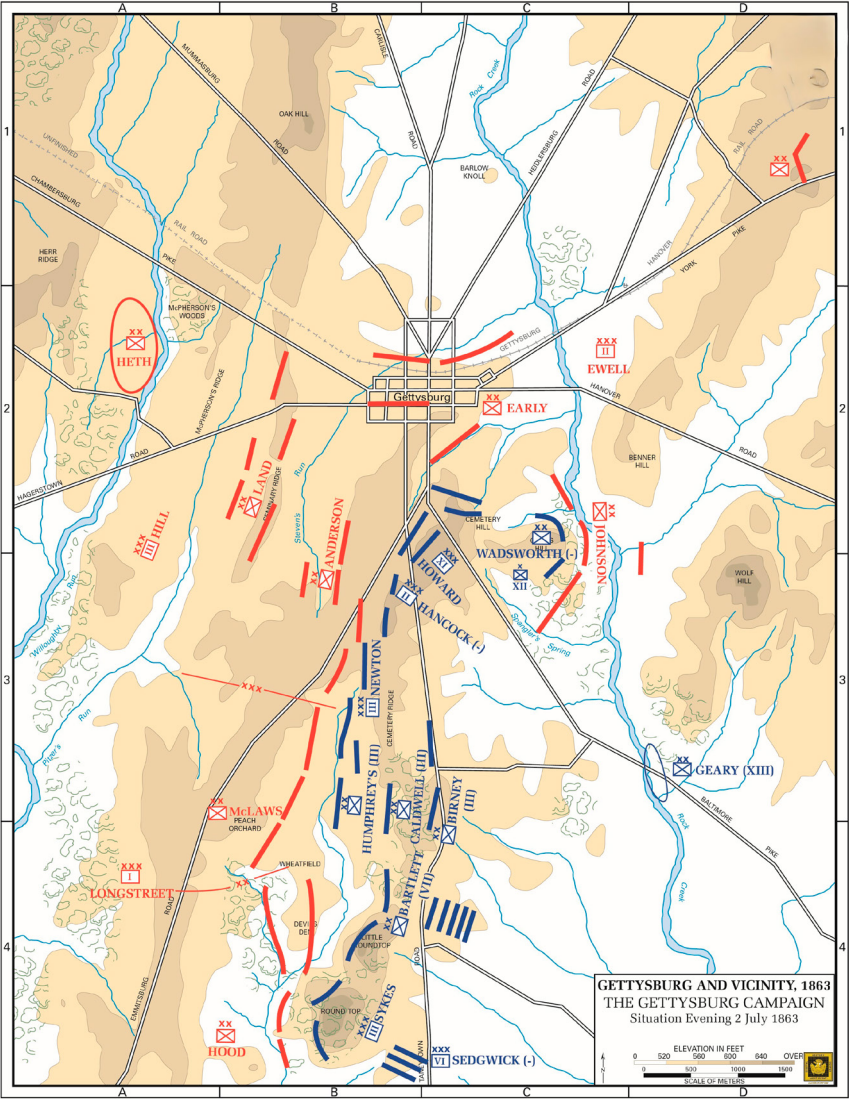
Lee made his decision during the night of July 1st that he would attack the Union forces the next day because he believed that Meade would not be able to get enough of his corps to the battle to stop the Confederate forces. Meade, with only five days in command, was willing to prepare his defense and wait for Lee to make a move. Shortly after noon on July 2nd, Meade's forces saw movement on the flanks of their defenses. Lee was on the attack.²⁹

Longstreet's corps broke through the forward Union lines and rolled up the defenses of the left side of the Union line. General Richard S. Ewell, attacking on the right side of the main Union defenses, was not successful against the well-established positions. By the end of the second day, Longstreet had given up some of the terrain he had gained against the Union left, and Ewell was unable to maintain any forward movement against the Union right.

During the evening of July 2nd, Meade held a meeting at his headquarters to discuss his strategy for the next day. With nine senior generals in his presence, there was a collective belief that the Union should continue to defend on Day 3. Meade, speaking to the commander of the Second Corps, the corps that would be the center of the Union defense, said, "If Lee attacks tomorrow, it will be '*in your front*' . . . Because he has made attacks on both our flanks and failed and if he concludes to try it again, it will be on our centre."³⁰

Lee, unhappy with the outcome of the day, sought no counsel from his subordinate generals. Instead, he decided to attack the

Gettysburg July 2



Map courtesy of the United States Military Academy Department of History & War Studies.

Image 5.6

Gettysburg July 3

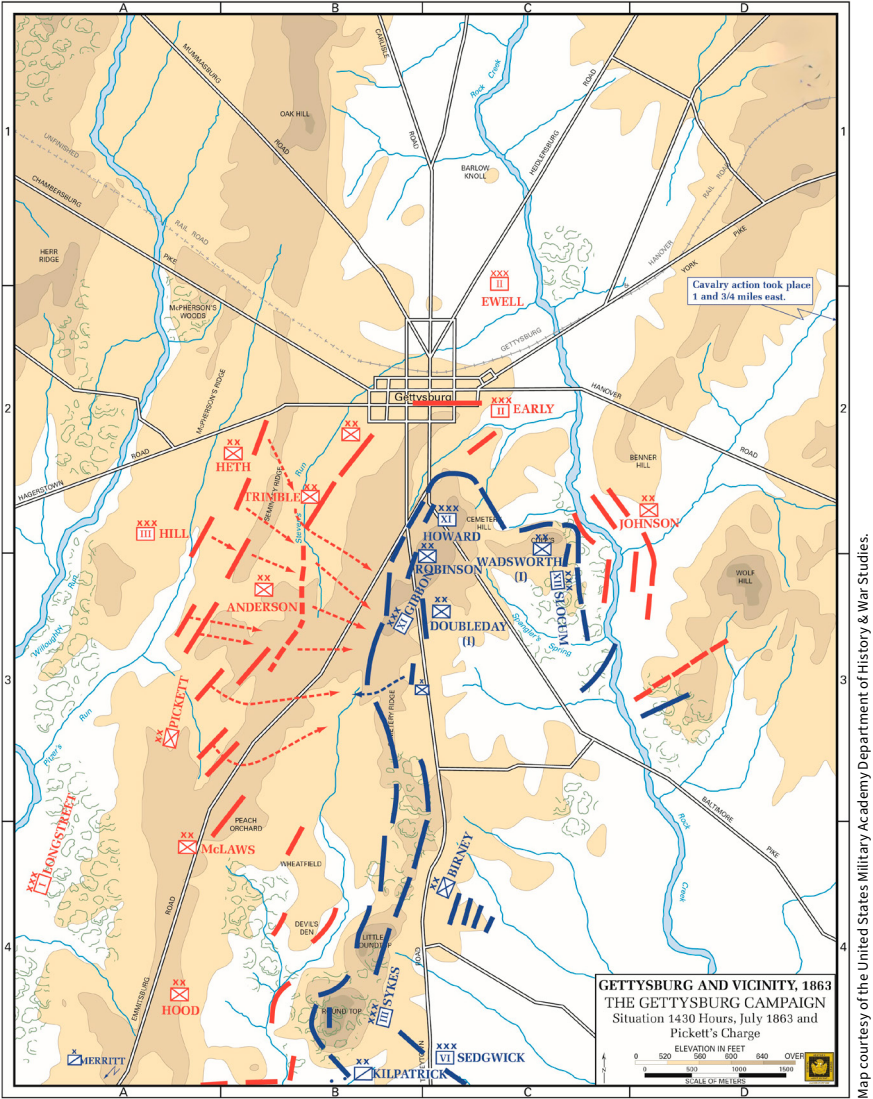


Image 5.7

Union defenses in the center. His orders, sent by courier to his corps commanders, were not received with the positive support reached by Meade's generals on his decision to defend. Longstreet and Ewell believed that an attack would meet with utter defeat.³¹ On July 3rd, their beliefs were substantiated when, in what would be remembered as Picket's Charge, 15,000 Confederate soldiers attacked the Union center on Cemetery Ridge and Ewell's forces continued the assault on Culp's Hill they had started the day before.

The end result was a resounding defeat for the Confederate forces. Having started the battle with a less than 10 percent deficit in men, Lee would finish the battle having suffered almost 40 percent casualties. Lee decided to retreat from Gettysburg and began heading south and west through Maryland toward northern Virginia on July 4th. Meade, suffering 27 percent casualties and maintaining a numeric superiority to Lee, did not seize the initiative and counterattack Lee as he retreated. His initial justification for heading to Gettysburg was "to mass the whole of the Army of the Potomac at Gettysburg to attack some fraction of the enemy's army."³² Now, given that opportunity, he chose to stay in his defenses.

II. The Game: *The Battle for Gettysburg*

As discussed in the previous chapters that outlined game construction, several things must be accumulated to define a game. The first is the level of analysis. Is this game going to be used to analyze the Battle for Gettysburg at the tactical or strategic level? Since the goal is to evaluate the fundamental strategic choices facing Lee and Meade, our game, which we will call *The Battle for Gettysburg*, will analyze the strategic decisions of these opposing generals. The second is the fundamental structure. Is this a sequential-play game in which Lee takes his turn then Meade responds, or is this a simultaneous-play game in which courses of action are chosen and then allowed to come together all at once? The historical narrative of the actual battle suggests a

sequentially played game. Finally, the players, the courses of action that may be chosen, as well as the players' outcomes, given all the possible course-of-action sequences, must be determined.

The game is going to start after the engagement on Day 1, when Meade believed two-thirds of Lee's forces were centered in Gettysburg. He was unsure of the location of Lee's First Corps and assumed he could engage and defeat the two Confederate corps with the bulk of his forces before Longstreet's corps could arrive on the field of battle. Meade pushed back during the first days' fighting, and the Union forces had occupied a key piece of high ground and began preparing their defenses. As the rest of Meade's forces arrived, they filled in the newly constructed defensive positions and made them stronger.

The game starts here. Lee has defended his position in Gettysburg and feels the momentum from that success. Meade has a defensible position and consolidating forces. This situation is best described by a sequential-play game in which Lee has the initiative of the first decision, with Meade left to respond.³³

Finding the Pieces

The Players of *The Battle for Gettysburg* As the scope of this game is to evaluate the fundamental strategic choices of Lee and Meade at Gettysburg, the strategic players in this game are self evident, being Lee, the commander of the Army of Northern Virginia, and Meade, the commander of the Army of the Potomac. As in the game *Operation Overlord/Fortress Europe*, there is a nonstrategic player as well: the randomness we called Nature.

Their Courses of Action for *The Battle for Gettysburg* While holding Gettysburg, Lee felt that he had the initiative. The initial victory solidified his belief that the Northern soldiers were no match for his forces. Additionally, he was comfortable with the information he had on the Northern troops' strength and location. Armed with the information that Meade was preparing defenses, Lee is placed in

the position of being the first mover in the game. As such, Lee is limited to basically three initial courses of action for the second day of fighting.

The first course of action for General Lee is to retire from Gettysburg. Although his forces had achieved initial success, Lee knew that the North was reinforcing its positions with numbers superior to his own. He can retire from the field and return to the Confederate States, having once again bloodied the Army of the Potomac's nose.

Lee's second course is to secure his position in Gettysburg. This option was discussed at length by Lee and his subordinate commanders. As Stephen W. Sears indicated in his book *Gettysburg*, Longstreet argued for a sweeping strategic move around the left of the enemy defenses "to get between Meade and Washington, thereby forcing Meade to do the attacking to regain his supply line and defend his capital."³⁴ If this was too ambitious a move, Longstreet offered a second but similar suggestion: a turning movement on the current battlefield, forcing Meade's soldiers out of their current defenses to attack Lee. Both tactical moves would give Lee the defensive advantage. Lee can secure Gettysburg, build up and use his defensive tactical advantage to, at the very least, show that the Army of Northern Virginia can hold federal territory, and possibly force the Army of the Potomac onto the offensive.

Lee's final course is to attack. If he can fight the Union forces before they are able to fully prepare defenses and receive their imminent reinforcements, Lee believes his army would prevail by destroying a significant portion of the Northern Army. He can take advantage of his momentum from the first day's victory and attack Meade, illustrating he has offensive capabilities. This first stage of the game, Lee's decision node and his available course-of-action choices, is illustrated in Figure 5.1.

General Lee's Courses of Action

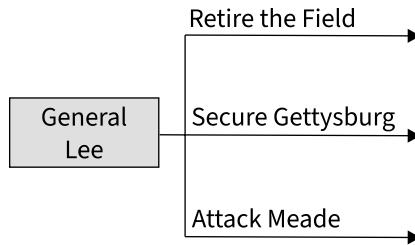


Figure 5.1: General Lee has three basic courses of action: retire from the field, secure Gettysburg, or attack General Meade.

Once Lee chooses a course of action, Meade must respond. Meade's responses are contingent on first-mover Lee's choices. Starting with Lee's top course of action in Figure 5.1, Lee retires from the field, Meade has three possible courses of action. Meade's course-of-action choices are shown in Figure 5.2.

Meade's first course of action, contingent on Lee's choice to retire from the field, is to reciprocate and also retire from the field. Since the threat of the Army of Northern Virginia penetrating farther toward Washington is gone, Meade knows he can retire the field and he would not risk opening an avenue for Lee to reach Washington. Additionally, given Lee's departure, Meade's choice to retire from the field would not adversely affect the fighting spirit of his army and would not cause Lincoln to relieve him of command.

Meade's second course of action is to secure Gettysburg, preventing Lee's return. This choice does tie Meade's forces or a portion thereof to Gettysburg, making them unavailable for a future fight.

Finally, Meade can pursue and attack the retreating Army of Northern Virginia. The choice to pursue Lee will bring Meade and Lee into direct conflict. We will discuss the resolution of battles when we discuss the role of the nonstrategic player, Nature.

General Meade’s Response to Retire Field

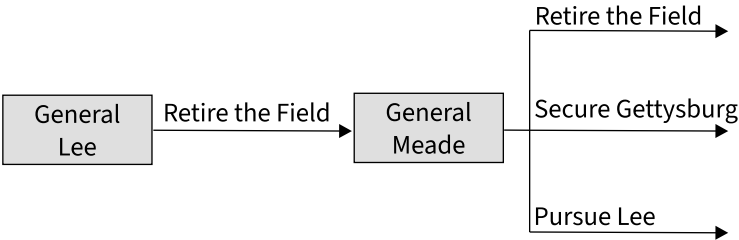


Figure 5.2: If General Lee retires the field, General Meade has three basic courses of action: retire the field, secure Gettysburg, or pursue and attack Lee.

Lee’s second course of action from Figure 5.1 is to defend and secure his position at Gettysburg. Given Lee’s choice to secure and hold Gettysburg, Meade has three possible courses of action. Meade’s choices are shown in Figure 5.3.

Meade’s first course-of-action choice is to retire from the field. This decision would leave the threat of the Army of Northern Virginia penetrating farther into Northern territory and possibly toward Washington. It would also adversely affect the fighting spirit of his army and would give Lincoln a good reason to relieve him of command.

Meade’s second course-of-action choice is to fortify his defensive position. This would prevent Lee from continuing farther into Northern territory or into Washington. It would also fix Lee’s forces in a precarious position when their supplies started to wane. Lee would be forced to scavenge for supplies in enemy territory, allowing Meade to attack the scavengers.

Meade’s third course-of-action choice is to attack Lee’s defensive position in Gettysburg. This choice would put his forces and Lee’s forces into direct conflict and bring in the nonstrategic player, Nature.

Lee’s third course of action from Figure 5.1 is to attack Meade. Given Lee’s choice to attack the Northern troops, Meade has three possible courses of action. Meade’s choices are shown in Figure 5.4.

General Meade's Response to Secure Gettysburg

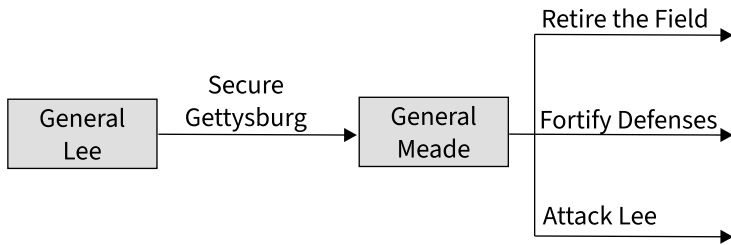


Figure 5.3: If General Lee secures Gettysburg, General Meade has three basic courses of action: retire the field, fortify his defenses, or attack Lee's position at Gettysburg.

General Meade's Response to Attack

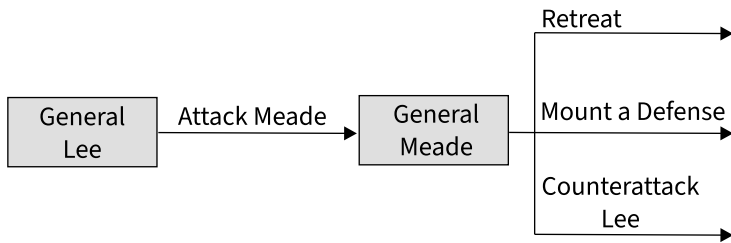


Figure 5.4: If General Lee chooses to attack Meade's position, General Meade has three basic courses of action: retreat, mount a defense, or counterattack Lee.

Meade's first course-of-action choice to a Southern attack is to retreat. As discussed earlier, this course-of-action choice would leave the threat of the Army of Northern Virginia penetrating farther into Northern territory and possibly toward Washington. It would also adversely affect the fighting spirit of his army and would again give Lincoln a reason to relieve him of command. In addition, a retreating Meade could be pursued by Lee's forces.

Meade's second course-of-action response to an attack is to mount a defense in his position outside of Gettysburg. This choice would cause Lee to fight against a defended position. This choice by Meade would pit his forces against Lee's in direct conflict and brings in the nonstrategic player, Nature.

Lastly, Meade could set up a counteroffensive and attack Lee. This choice would again put his forces and Lee's forces into direct conflict and bring in the nonstrategic player, Nature.

Finally, we discuss the course-of-action response an attacking general has when his opponent chooses to retreat. In the example above, Lee could choose to attack Meade's position outside of Gettysburg. In response, Meade could choose to retreat. If Meade retreats, Lee now has a decision node, as shown in Figure 5.5. The attacking general, in this case, Lee, has two course-of-action choices, if the attacked general, in this case, Meade, chooses to retreat.

Lee's first choice is to let Meade retreat without pursuit. At this point, no further conflict will ensue and the game tree will resolve to its outcomes.

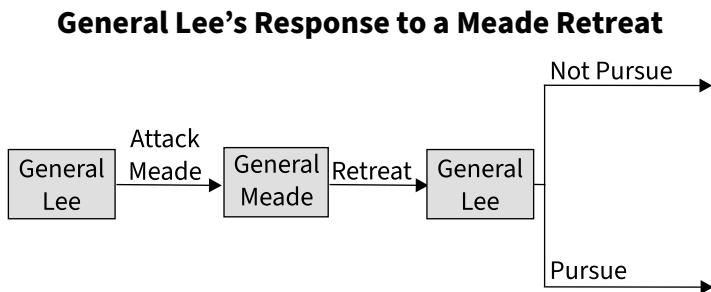


Figure 5.5: With General Meade in retreat, General Lee can choose to pursue or to not pursue. If Lee chooses to pursue, Nature will resolve the battle in one of two ways: a Lee victory, or a Lee failure, in which Meade escapes relatively unharmed. If Lee chooses to not pursue, Meade will successfully escape.

Lee's second course-of-action choice is to pursue the retreating Meade to inflict damage to the retreating army. This choice by Lee would put the forces into conflict and bring in the nonstrategic player, Nature. Nature only enters the game when the Army of the Potomac and the Army of Northern Virginia come to battle. There are three fundamental ways that battle is engaged in this game.

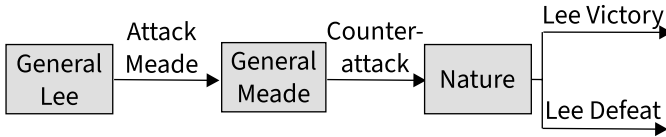
The first is when both armies choose to take an offensive posture. An example is the set of choices in which Lee attacks Meade and Meade chooses to counterattack. Under this circumstance, from the perspective of the player with the first move, Lee, Nature will resolve the battle with one of two outcomes. Lee will win the battle, defeating Meade, or Meade will win the battle, defeating Lee. An example of Nature's available courses of action is the top game tree illustrated in Figure 5.6.

The second direct attack form occurs when one of the armies takes a defensive posture and is attacked by the other. An example is the set of choices in which Lee attacks Meade and Meade chooses to defend his position. Under this circumstance, from the perspective of the player with the first move, Lee, Nature will resolve the battle with one of two outcomes. Lee will win the battle by overcoming Meade's defense, or Meade wins by having defended his position. An example of Nature's available courses of action is shown in the second game tree illustrated in Figure 5.6.

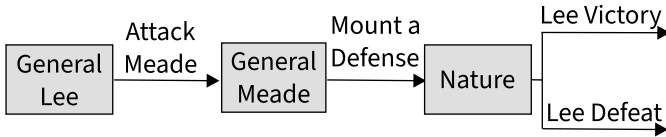
The third type of battle occurs when one army chooses to retreat and the other army chooses to pursue. As an example, consider Lee attacking Meade and Meade choosing to retreat. Under these conditions, Lee has another decision node and can press his attack and pursue while Meade retreats or Lee can choose not to pursue Meade and consolidate his gains. Figure 5.6 illustrates the potential outcomes given Nature's intervention in the case when Lee chooses to pursue: either a Lee victory, or Lee failing to prevent Meade's escape in the third game tree.

Nature's Response to Direct Attacks

Both Armies Take an Offensive Posture



One of the Armies Takes a Defensive Posture



One of the Armies Retreats

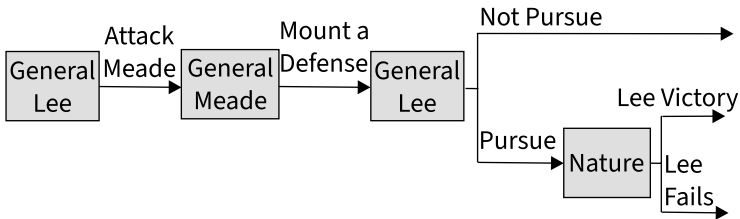


Figure 5.6: If the two armies engage in battle in which both take an offensive posture, Nature will resolve the battle in one of two ways: a Lee victory, or a Lee defeat. If one of the two armies takes a defensive posture, then nature will resolve in one of two ways: a Lee victory or a Lee defeat. If one of the two armies retreats from a direct attack and the opposing army chooses to pursue, then Nature will resolve the battle in one of two ways: a Lee victory or a Lee failure in which the retreating army successfully escapes.

All the course-of-action choice sets for Lee and Meade are assembled in Figure 5.7. To build the complete extensive form of the sequential-play game *The Battle for Gettysburg*, each player's respective outcomes must be determined.

The Battle for Gettysburg

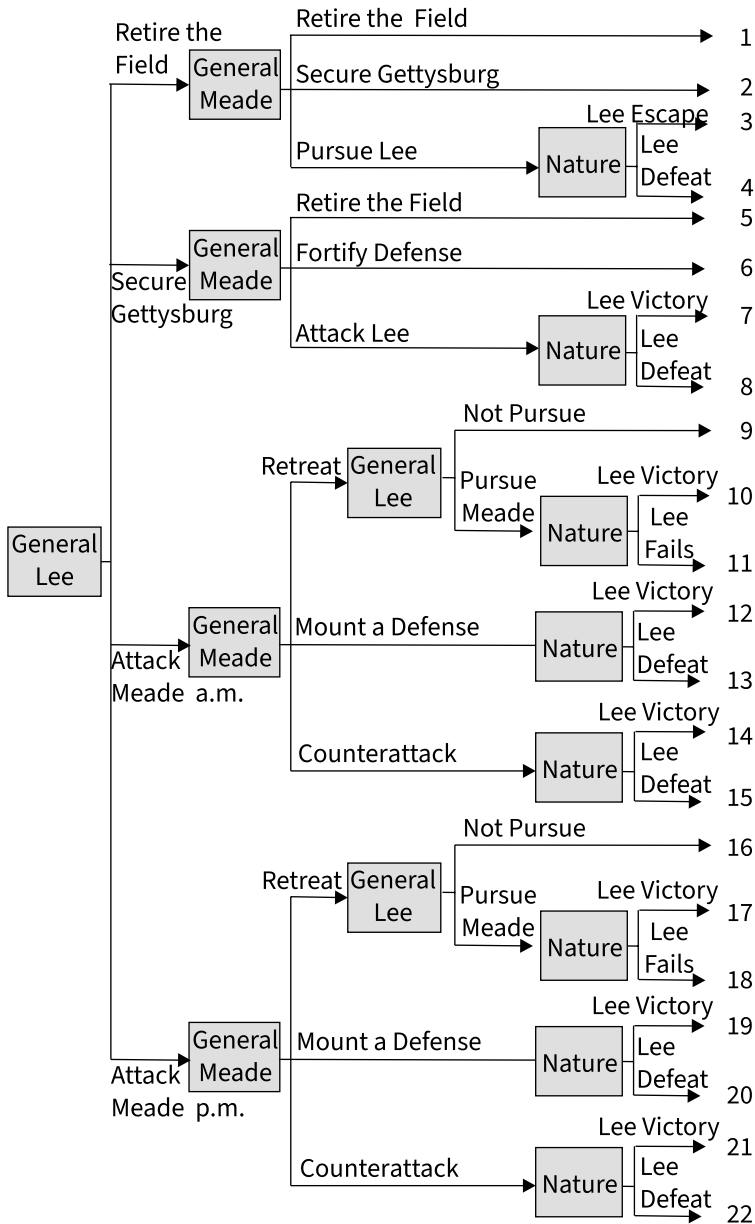


Figure 5.7: The course-of-action sequences in *The Battle for Gettysburg* resolve twenty-two possible game outcomes.

Outcomes for *The Battle for Gettysburg* Having built up the basic flow of *The Battle for Gettysburg*, from determining the fundamental structure of the game as sequential play, to delineating the players, Lee, Meade, and Nature, and their potential courses of action, all that is left to determine, in order to have a fully defined game, is the set of contingent outcomes available to each player.

Figure 5.7 lays out the extensive form of *The Battle for Gettysburg* and enumerates the twenty-two possible course-of-action sequences that lead to outcomes for this game. Of the twenty-two sequences, six of them do not result in direct conflict between Lee and Meade. In the other sixteen sequences, Lee and Meade do engage in direct conflict, and the nonstrategic player, Nature, resolves the battle to its final outcome. The course-of-action sequences that lead to battle fall into three fundamental forms shown in Table 5.1.

Table 5.1: Battle-Contingent Outcomes

BATTLE FORM	COURSE-OF-ACTION SEQUENCE
Attack in Pursuit	Lee retires the field Meade pursues
	Lee attacks Meade retreats Lee pursues
Both Attack	Lee attacks Meade counterattacks
Attack Defended Position	Lee secures Gettysburg Meade attacks
	Meade attacks Meade defends

Nature resolves an attack in pursuit into two potential outcomes. The pursuer either defeats the one in retreat, or the one in retreat escapes. In the case of Lee retiring the field and Meade pursuing, then from the perspective of Lee, Lee either evades or is defeated. In either outcome, Meade has possession of Gettysburg. If Lee is defeated, then Meade has Gettysburg and has defeated Lee. In the case of Lee attacking, Meade retreating, and Lee pursuing, Nature resolves the battle to a Lee victory or a Lee failure. To Meade, this would correspond to a Meade loss or a Meade escape. A battle defeat to either general is of course the worst possible outcome in the game, while a battle victory is the best possible outcome.

In the case in which Lee attacks, either in the a.m. or the p.m., and Meade counterattacks, Nature resolves the battle to a Lee victory or a Lee defeat. The complementary outcomes for Meade are his loss or a victory. The cases of attacking on a defended position resolve the same as in attack/counterattack cases. Nature resolves the battle to a Lee victory or a Lee defeat. As indicated in the previous paragraph, a battle defeat to either general is the worst of possible outcomes, and a battle victory is the best of possible outcomes.

There remain six potential course-of-action sequences, as illustrated in Table 5.2. In these six course-of-action sequences, Meade and Lee do not engage in battle. Course-of-action sequence one, in which both Lee and Meade retire from the field, ends with no additional battle. Lee has shown some offensive prowess but has no definitive victory, Meade has neither won nor lost, but the South has withdrawn. This is not a satisfactory outcome for the Confederacy as definitive offensive prowess must be shown to secure external funding for the war effort. This outcome is better than a loss but is less preferred than a neutral outcome. As to the United States, this is not a victory, but it is a good outcome nonetheless. The outcome to course of action sequence two is only different from one in that Meade has forces tied up in Gettysburg, performing a task better left to the local militia.

Table 5.2: Battle-Free Outcomes

OUTCOME NUMBER	COURSE-OF-ACTION SEQUENCE
1	Lee retires the field Meade retires the field
2	Lee retires the field Meade secures Gettysburg
5	Lee secures Gettysburg Meade retires the field
6	Lee secures Gettysburg Meade fortifies to block
9	Lee attacks a.m. Meade retreats Lee does not pursue
16	Lee attacks p.m. Meade retreats Lee does not pursue

Course-of-action sequences five and six involve Lee fortifying his position in Gettysburg. Historically, Lee dismissed the course of action to secure and defend Gettysburg because he was concerned that remaining in defense beyond his week’s supply of food, waiting for a potential Meade attack, would mean, as noted by Stephen Sears in his book *Gettysburg*, Lee’s forces would have to forage in the countryside,³⁵ subjecting his forces to serious risk due to the presence of local Union militia. If Meade chose to fortify and defend in the face of Lee’s defenses, no battle would follow, and the result would be that the South remained on Northern soil but no decisive engagement would have taken place. As discussed earlier, Meade knew he could not retire from the field without risking opening an avenue for Lee to reach Washington. Additionally, retiring the field under these

circumstances would hurt the “esprit” of his army as well as cause Lincoln to relieve him of command. The course-of-action sequences leading to game outcomes are indicated in Table 5.3.

Given the outcomes listed in Table 5.3, the extensive form of the sequential game *The Battle for Gettysburg* is illustrated in Figure 5.8. The twenty-two course-of-action sequences are laid out to the outcomes from Lee’s perspective. The corresponding Meade outcomes, listed in Table 5.3, are not included in Figure 5.8 for the purposes of clarity.

Table 5.3: COA Sequences and Associated Outcomes

	PLAYER COA SEQUENCES	NATURE	OUTCOMES
1	Lee Retires Field, Meade Retires Field		Lee Leaves the North Meade Leaves Gtybg
2	Lee Retires Field, Meade Secures Gettysburg		Lee Leaves the North Meade Holds Gtybg
3	Lee Retires Field, Meade Pursues Lee	Lee Evades	Lee Leaves the North Meade Holds Gtybg
4	Lee Retires Field, Meade Pursues Lee	Lee Defeat	Lee Defeated Meade Victory
5	Lee Secures Gettysburg, Meade Retires Field		Lee Gtybg Unopposed Meade Leaves Gtybg
6	Lee Secures Gettysburg, Meade Fortifies Defense		Lee Gtybg Opposed Meade Stop Incursion
7	Lee Secures Gettysburg, Meade Attacks Lee	Lee Victory	Lee Victory Meade Defeat
8	Lee Secures Gettysburg, Meade Attacks Lee	Lee Defeat	Lee Defeat Meade Victory
9	Lee Attacks Meade a.m., Meade Retreats, Lee Does Not Pursue		Lee Gtybg Unopposed Meade Leaves Gtybg

(continued)

Table 5.3 *(continued)*

	PLAYER COA SEQUENCES	NATURE	OUTCOMES
10	Lee Attacks Meade a.m., Meade Retreats, Lee Pursues	Lee Victory	Lee Victory Meade Defeat
11	Lee Attacks Meade a.m., Meade Retreats, Lee Pursues	Lee Fails	Lee Gtybg Unopposed Meade Leaves Gtybg
12	Lee Attacks Meade a.m., Meade Defends Position	Lee Victory	Lee Victory Meade Defeat
13	Lee Attacks Meade a.m., Meade Defends Position	Lee Defeat	Lee Defeat Meade Victory
14	Lee Attacks Meade a.m., Meade Counterattacks	Lee Victory	Lee Victory Meade Defeat
15	Lee Attacks Meade a.m., Meade Counterattacks	Lee Defeat	Lee Defeat Meade Victory
16	Lee Attacks Meade p.m., Meade Retreats, Lee Does Not Pursue		Lee Gtybg Unopposed Meade Leaves Gtybg
17	Lee Attacks Meade p.m., Meade Retreats, Lee Pursues	Lee Victory	Lee Victory Meade Defeat
18	Lee Attacks Meade p.m., Meade Retreats, Lee Pursues	Lee Fails	Lee Gtybg Unopposed Meade Leaves Gtybg
19	Lee Attacks Meade p.m., Meade Defends Position	Lee Victory	Lee Victory Meade Defeat
20	Lee Attacks Meade p.m., Meade Defends Position	Lee Defeat	Lee Defeat Meade Victory
21	Lee Attacks Meade p.m., Meade Counterattacks	Lee Victory	Lee Victory Meade Defeat
22	Lee Attacks Meade p.m., Meade Counterattacks	Lee Defeat	Lee Defeat Meade Victory

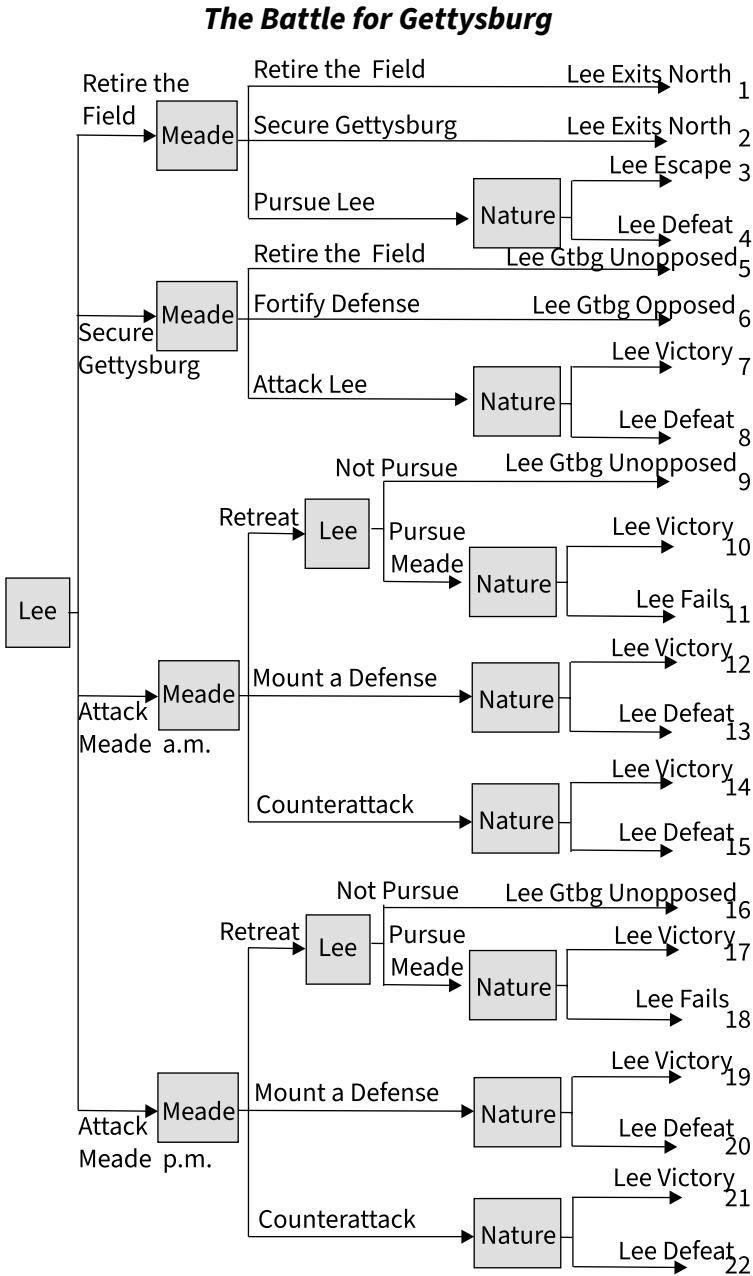


Figure 5.8: *The Battle for Gettysburg* outcomes delineated from the perspective of General Lee. See Table 5.3 for General Meade's corresponding outcomes.

Outcome Valuations To complete *The Battle for Gettysburg* to a form that can be solved for a Nash equilibrium, the set of outcomes given in Table 5.3 needs to be associated with a set of preference orderings. For the purposes of this case, we will make some simplifying assumptions about the preference orderings Lee and Meade have over the outcomes listed. These simplifications make the case easier to follow. For a more accurate portrayal, a much more nuanced outcome set and preference ordering can be outlined.

For Lee, we are assuming victory in battle with Meade is preferred to holding Gettysburg unopposed, which is preferred to holding Gettysburg opposed, which is preferred to being defeated in battle by Meade. For Meade, we are assuming that victory in battle with Lee is preferred to holding Gettysburg, which is preferred to leaving Gettysburg, which is preferred to being defeated in battle by Lee.

Ordinal preference rankings for Lee and Meade are given outcome valuations ranging from zero, the least preferred outcome, to four, the most preferred outcome. As discussed above, if there is a battle, a victory will be valued at four and a defeat will be valued at zero. The other outcomes will have outcome valuations according to their position in the respective generals' preference orderings. The full matching of outcomes to outcome valuations in descending order is delineated in Table 5.4 and in Figure 5.9.

Table 5.4: Outcomes and Outcome Valuations

GENERAL LEE		GENERAL MEADE	
Outcome	Value	Outcome	Value
Victory	4	Victory	4
Gtbg Unopposed	3	Holds Gtbg	3
Gtbg Opposed	2	Defends Incursion	2
Leave the North	1	Leaves Lee Unopposed	1
Defeat	0	Defeat	0

In a real wartime scenario, it is easy to assume victories and defeats have different preference orderings contingent on the context of each outcome, the severity of the defeat, or the scale of the victory. For simplicity's sake and for illustrative purposes, these considerations are ignored in this game.

Given the outcome preference valuations, the next step to solving the game, *The Battle for Gettysburg*, the unknowable course-of-action choice of the nonstrategic player Nature must be resolved.

Since Nature is a stochastic or random player, the most information we can get about Nature's choices is the probability distribution across its choice set. That is, we must determine the probability of Nature resolving to any of three either-or scenarios: a Lee victory or a Lee defeat in direct battle scenarios; a Lee victory or Lee in Gettysburg unopposed when Lee pursues a retreating Meade; and Lee leaving the North or a Lee defeat.

Nature's Probability Distributions Referring to Figure 5.9, there are seven battle scenarios where the nonstrategic and stochastic player Nature resolves to outcomes for Lee and Meade in the game. To consider the probability distribution across Nature's resolution to outcomes of direct battle in an open field, where victory or defeat are the only potential outcomes, there are assumptions that must be made. There are three fundamental situations under which battle can occur in *The Battle for Gettysburg*.

The first is a mutual attack. The foundational assumption is that for equally matched forces in terrain, manpower, resources, and experience, either force has a probability of victory of 50 percent, or $\rho_{\text{victory}} = 0.50$. Asymmetry in any of the matching assumptions (manpower differences, resource differences, etc.) will change the probability of victory one way or the other.

The second is an attack on a defended position. The foundational assumption for attacking a fortified defense is that an attacker with a three-to-one force advantage has a probability of victory of 50 percent, or $\rho_{\text{victory}} = 0.50$.³⁶ Again, if the three-to-one force-advantage rule

The Battle for Gettysburg

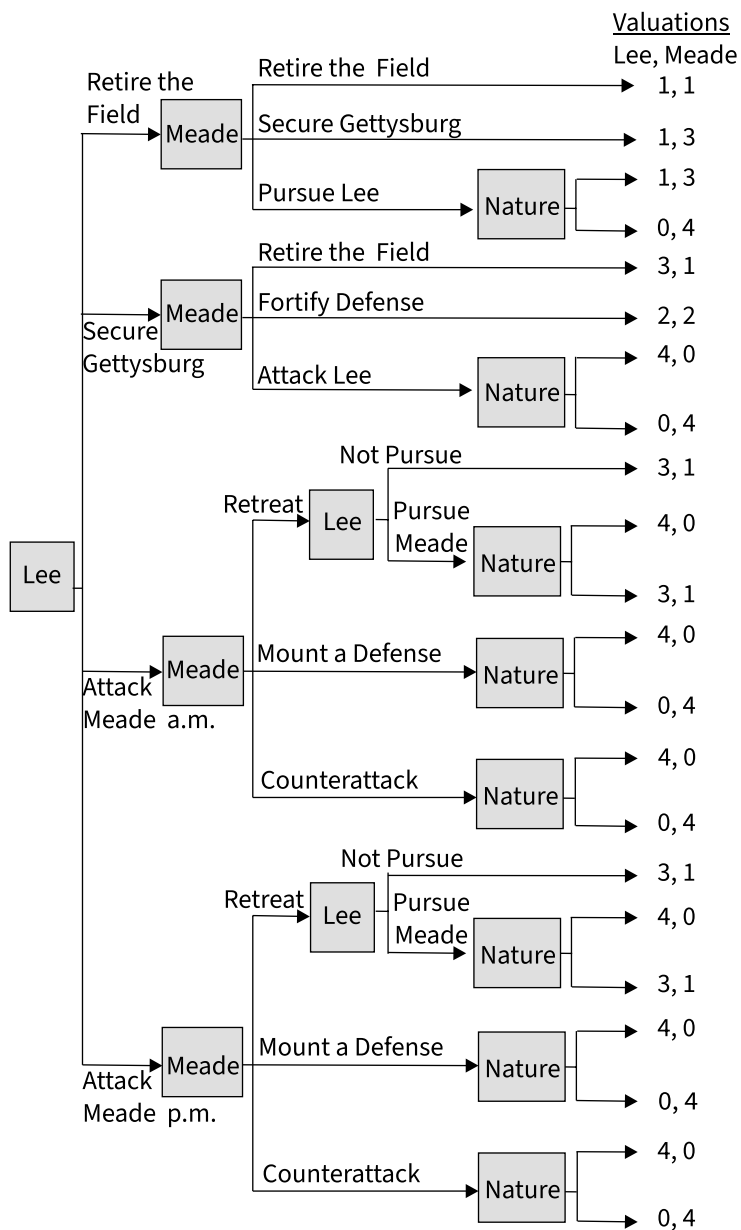


Figure 5.9: *The Battle for Gettysburg* with General Lee's and General Meade's outcome valuations from Table 5.4.

is not met in the actual battle scenario, the probability of victory will be changed one way or the other.

The third is a battle while in pursuit. Under these circumstances, we assume that it is the retreating force's overwhelming advantage to escape the pursuit. The pursuing force can cause the retreating force significant damage, but to turn this situation into a battle is extremely rare. For the sake of this exercise, the probability of the pursued force escaping is 90 percent or $p_{\text{escape}} = 0.90$. These probabilities are reported in Table 5.5 for outcomes three and four, ten and eleven, and seventeen and eighteen, and are presented in Figure 5.10.

In order to quantify the probability of success for each of the potential confrontations between Lee and Meade's forces, the numeric superiority of the North and the perceived fighting superiority of the South must be considered. The official figures, provided by the Gettysburg Welcome Center,³⁷ indicate that Lee had 75,000 men and Meade had 83,000 men under their commands.

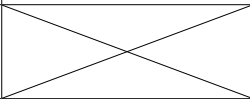
After the first day of battle, Lee assumed Meade would only have 60 percent of his infantry corps available to fight at Gettysburg on the morning of Day 2.³⁸ Since Lee had all three of his corps available at Gettysburg, he would have a numeric advantage of 75,000 to Meade's 50,000 (Meade's forces calculated as $83,000 \times .6$, rounded up.³⁹) Given that Lee assumed Meade would have his remaining corps on the move and be able to close on Gettysburg within eighteen hours, Lee assumed this advantage was only true for the morning of Day 2; later on in the day the remaining Union corps would arrive at Gettysburg.⁴⁰ Once the remaining Union forces had arrived, Lee's advantage would be reversed so that the Union had 83,000 and Lee had 75,000. However, because Lee believed his men were more experienced fighters than the Union soldiers, Lee calculated that he would have combative parity with Meade's larger force. Therefore, Lee believed he had a 1.5:1 force advantage for a morning battle but would have a 1:1 parity if the battle did not take place until the afternoon of the second day.

Table 5.5: Outcomes, Valuations, and Probabilities

The twenty-two possible outcomes from the game *The Battle for Gettysburg* and links to their corresponding valuations to both General Lee and General Meade. The conditional probabilities associated with Nature’s resolution of the conflict outcomes are included in the final column.

		OUTCOME VALUATION		PROBABILITY
	Outcome	Lee	Meade	Lee
1	Lee Leaves the North Meade Leaves Gtybg	1	1	
2	Lee Leaves the North Meade Holds Gtybg	1	3	
3	Lee Leaves the North Meade Holds Gtybg	1	3	$\rho_{\text{Escape}} = 0.90$
4	Lee Defeated Meade Victory	0	4	$\rho_{\text{Defeat}} = 0.10$
5	Lee Gtybg Unopposed Meade Leaves Gtybg	3	1	
6	Lee Gtybg Opposed Meade Stop Incursion	2	2	
7	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.80$
8	Lee Defeat Meade Victory	0	4	$\rho_{\text{Defeat}} = 0.20$
9	Lee Gtybg Unopposed Meade Leaves Gtybg	3	1	
10	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.10$

Table 5.5 (*continued*)

		OUTCOME VALUATION		PROBABILITY
11	Lee Gtybg Unopposed Meade Leaves Gtybg	3	1	$\rho_{\text{Fail}} = 0.90$
12	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.25$
13	Lee Defeat Meade Victory	0	4	$\rho_{\text{Defeat}} = 0.75$
14	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.60$
15	Lee Defeat Meade Victory	0	4	$\rho_{\text{Defeat}} = 0.40$
16	Lee Gtybg Unopposed Meade Leaves Gtybg	3	1	
17	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.10$
18	Lee Gtybg Unopposed Meade Leaves Gtybg	3	1	$\rho_{\text{Fail}} = 0.90$
19	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.15$
20	Lee Defeat Meade Victory	0	4	$\rho_{\text{Defeat}} = 0.85$
21	Lee Victory Meade Defeat	4	0	$\rho_{\text{Victory}} = 0.45$
22	Lee Defeat Meade Victory	0	4	$\rho_{\text{Defeat}} = 0.55$

Meade believed he would have his entire army massed in a defense by noon on Day 2.⁴¹ The information provided to him led him to believe Lee had at least two-thirds of his forces at Gettysburg, and Meade was not sure of the location of the other corps. Therefore, we infer that Meade believed he would, at best, have a 1.66:1 advantage and a 1.1:1 advantage at worst (if Lee received his third and final corps before the battle was over). A critical caveat on these relative force ratios is that, while no battle will have 100 percent of the forces from each side directly opposed, both sides will have the same *relative* number of soldiers engaged in battle at the same time. Therefore, the total number of soldiers available will be used to convey the planning of the commanders so as not to get lost in the specific numbers for each individual engagement across the battlefield.

As we can see, in the real battle, neither Lee nor Meade had perfect information on his opponent's force composition or readiness. But each general had very good information about his own force's disposition. For this game, we will assume that the nonstrategic player, Nature, responds to the actual force composition of each side and that Lee and Meade did as well. (In later chapters, we will explore the relevance of asymmetric information and how it affects player choices and game equilibria.)

The information just presented is sufficient to determine Lee's probabilities of victory or defeat in the direct battles defined by outcomes seven and eight, twelve and thirteen, fourteen and fifteen, nineteen and twenty, and twenty-one and twenty-two in Table 5.5 and presented in Figure 5.10.

If Lee secures his position in Gettysburg and Meade chooses to attack in the afternoon after his forces are complete, then the three-to-one rule for attacking a fortified defense applies. Meade needs a 3:1 force advantage for a 50 percent or probability of victory. Meade only would have a 1.1:1 force advantage, given a simple linear relationship between force size and probability of victory in an engagement. At best Meade has a 20 percent probability or $\rho_{\text{Defeat}} = 0.20$ of a Lee defeat (Meade victory) and thus a $\rho_{\text{Victory}} = 0.80$ probability of a Lee victory.

These probabilities are reported for outcomes seven and eight in Table 4.5 and presented in Figure 5.10.

If Lee chooses to attack in the morning and Meade chooses to defend his position, then Lee needs a 3:1 force advantage for a 50 percent or $\rho_{\text{victory}} = 0.50$ probability of victory. Lee's 1.50:1 force-size advantage translates to a 25 percent or $\rho_{\text{Victory}} = 0.25$ probability of a Lee victory and a $\rho_{\text{defeat}} = 0.75$. These probabilities are reported for outcomes twelve and thirteen in Table 4.5 and presented in Figure 5.10.

If Lee chooses to attack in the morning and Meade chooses to counterattack, where a baseline 1:1 force size equates to a 50 percent probability of victory or $\rho_{\text{victory}} = 0.50$, then a 1.50:1 force-size advantage for Lee translates to a 60 percent probability of victory $\rho_{\text{Victory}} = 0.60$ for Lee. This implies approximately a 40 percent loss probability or $\rho_{\text{loss}} = 0.40$ for Lee. These probabilities are reported for outcomes fourteen and fifteen in Table 5.5 and presented in Figure 5.10.

If Lee waits until the afternoon to attack, Meade will have gained his full force structure. Thus, an afternoon attack in which Meade chooses to defend would ensue whereby Lee would have a 0.9:1 force disadvantage to Meade. Lee has less than one-third of the forces necessary to meet the three-to-one rule for a 50 percent probability of victory attacking a defensive position. Given Lee's force disadvantage, he has approximately a 15 percent or $\rho_{\text{Victory}} = 0.15$ probability of victory and thus, $\rho_{\text{Defeat}} = 0.85$ probability of defeat. These probabilities are reported for outcomes nineteen and twenty in Table 5.5 and presented in Figure 5.10.

In the final battle scenario, Lee attacks in the afternoon and Meade counterattacks. As in the previous discussion, Lee faces a 0.9:1 force disadvantage in the afternoon after Meade receives the rest of his forces. Given the one-to-one rule for a 50 percent probability of victory, Lee only has a 45 percent or $\rho_{\text{Victory}} = 0.45$ probability of victory over Meade and thus, a 65 percent or $\rho_{\text{Defeat}} = 0.65$ probability of defeat. These probabilities are reported for outcomes twenty-one and twenty-two in Table 5.5 and presented in Figure 5.10.

Gettysburg: Valuations and Probabilities

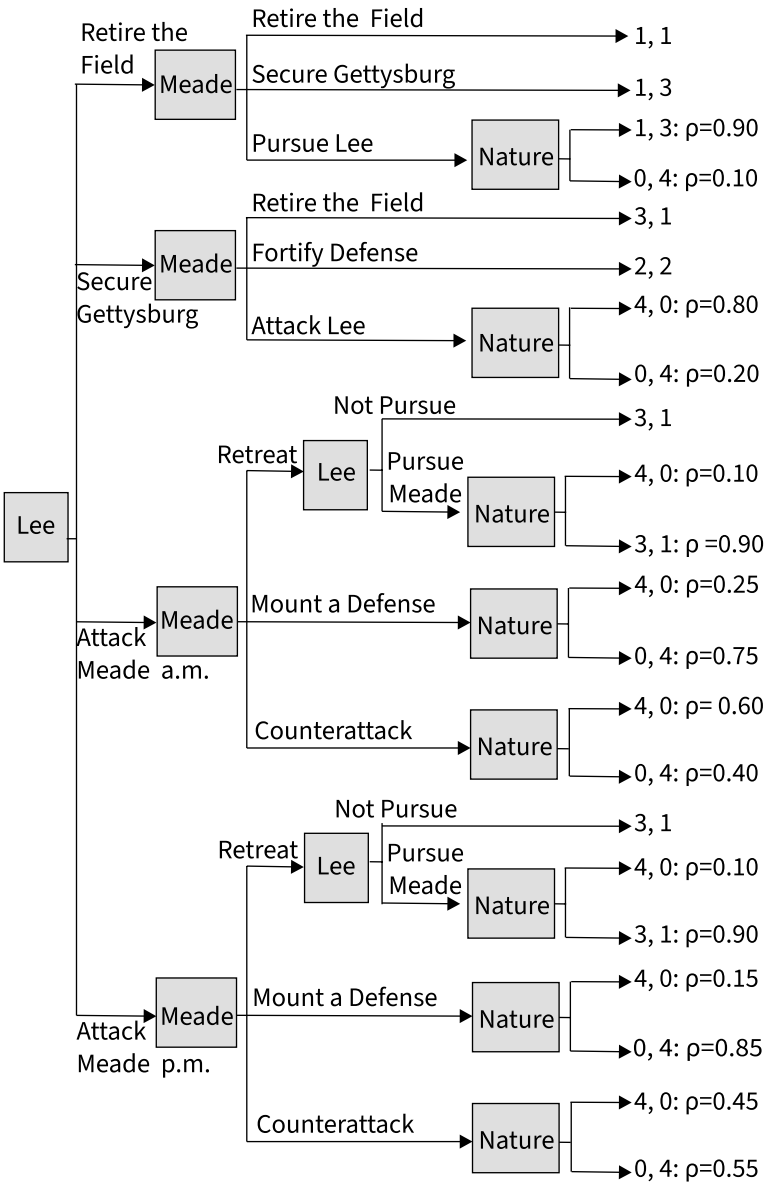


Figure 5.10: The battle sequences with General Lee’s and General Meade’s outcome valuations and the probabilities of nature resolving to specific battle outcomes.

The Battle for Gettysburg is now a fully defined game. Each course-of-action sequence of both strategic players, Lee and Meade, as well as the battle outcomes determined by the nonstrategic player, Nature, leads to well-defined outcomes. The outcomes are all given a preference ranking by both Lee and Meade. The uncertain outcomes determined by Nature have a well-defined probability ranking. Figure 5.10 is a complete representation of the game.

Solving the Game: *The Battle for Gettysburg* The next task we have is to resolve the game back to its sub-game perfect Nash equilibrium. This is accomplished the same way *Operation Overlord/Fortress Europe* was solved in Chapter III. We start with the outcome valuations and work our way back to Lee's first course-of-action choice. The first decision nodes we examine are the ones associated with the probabilistic player, Nature.

The first decision node for Nature leads to outcomes numbered three and four. This follows the course-of-action sequence in which Lee chooses to retire from Gettysburg and Meade chooses to pursue. Nature then resolves the game to outcome three or outcome four. In outcome three, Lee leaves the North while pursued by Meade but Lee escapes Meade's pursuit. This outcome happens with a probability $\rho_{\text{Escape}} = 0.90$ and an outcome valuation of one to Lee and an outcome valuation of three to Meade. In outcome four, Lee is defeated by Meade attempting to retire from Gettysburg. This outcome happens with a probability $\rho_{\text{Defeat}} = 0.10$ and an outcome valuation of 0 to Lee and an outcome valuation of 4 to Meade.

Neither Lee nor Meade has a perfect knowledge of how Nature will resolve the course-of-action sequence if Lee chooses to retire and if Meade chooses to pursue. They do have either a historic perspective or a doctrinal perspective on how this course-of-action sequence will turn out. Given that both understand the probability distribution across Nature's outcome choices, the generals can have an expectation regarding how Nature's choice will play out. So they both can calculate their expected outcome valuations. Lee knows that with a probability

of 0.90 he will escape Meade's pursuit and with a probability of 0.10 Meade will defeat him. Thus, Lee's expected outcome valuation is: $EV_{Lee} = 0.90 \times 1 + 0.10 \times 0 = 0.90$. Using the same logic for Meade, his expected outcome valuation is: $EV_{Meade} = 0.90 \times 3 + 0.10 \times 4 = 3.10$.

Taking Nature's probability distribution and applying it each time Nature plays a role in resolving a course-of-action sequence to outcomes, Figure 5.10 can be reduced to just the game between Lee and Meade as seen in Figure 5.11.

Having the probabilistic battle scenarios reduced to expected outcome valuations, we look at the next higher layer of decision-making. In this game, it is the Lee decision nodes determining whether to pursue a retreating Meade. In both cases, Lee is deciding between not pursuing, for an outcome valuation of three, or pursuing, for an expected outcome valuation of 3.10. Since the expected outcome valuation of choosing to pursue exceeds the outcome valuation of choosing to not pursue, a rational Lee would obviously choose to pursue his opponent.

Having resolved Lee's decision, in the case of Meade's retreat, the game analysis moves up one level to Meade's four decision nodes. Meade, knowing Lee's best response is to pursue, allows Meade to see his outcome valuations associated with any of his own course-of-action choices. Meade's interpretation of the way the game will play forward from his course-of-action choices are represented in Figure 5.12.

Meade will find himself at one of his four decision nodes. The one where Lee decided to retire the field; the one where Lee decided to secure Gettysburg; the one where Lee chooses to attack in the morning; and the one where Lee chooses to attack in the afternoon.

At the first decision node, Meade has three course-of-action choices: retire from the field and receive an outcome valued at three; secure Gettysburg and receive an outcome valued at three; pursue Lee's retiring force and receive an expected outcome valued at 3.10.

Meade, being a rational actor, will make the course-of-action choice that gives him the maximum outcome valuation. At this particular node, contingent upon Lee retiring from the field, Meade will pursue the retiring Lee for the expected outcome valuation of 3.10.

Gettysburg: Nature Expected Valuations

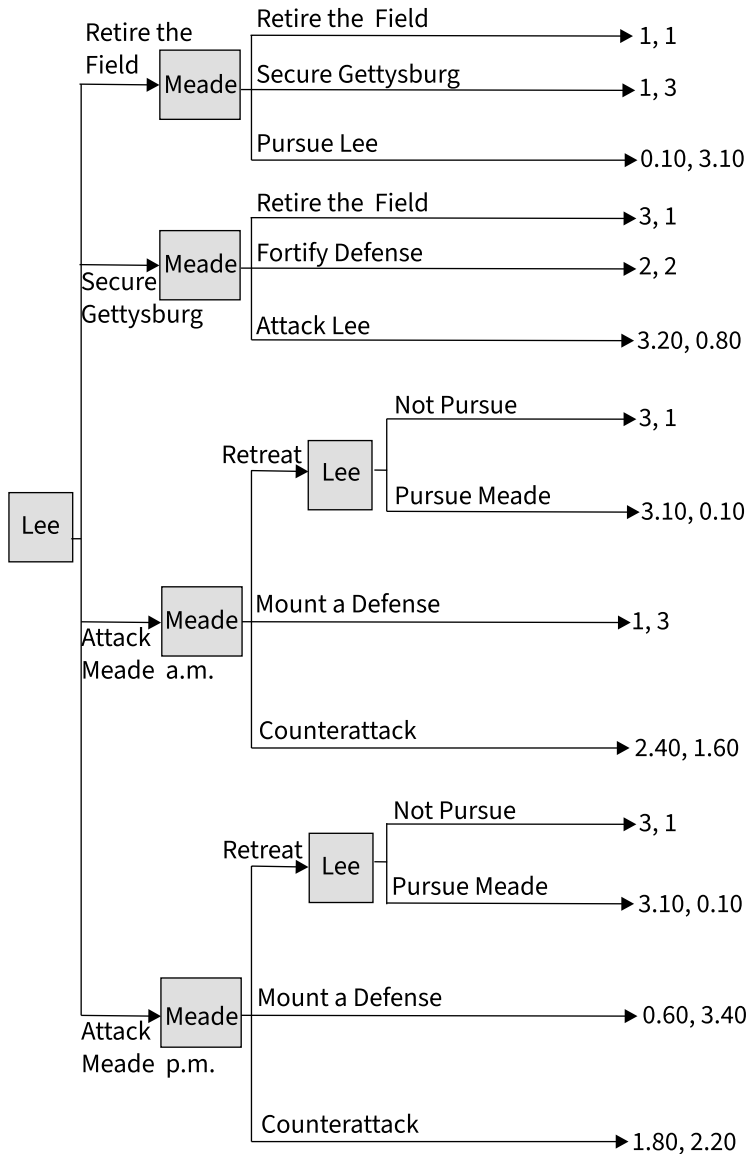


Figure 5.11: *The Battle for Gettysburg* with General Lee's and General Meade's expected outcome valuations, given Nature's probability distributions.

Gettysburg: Lee's Sub-Games Resolved

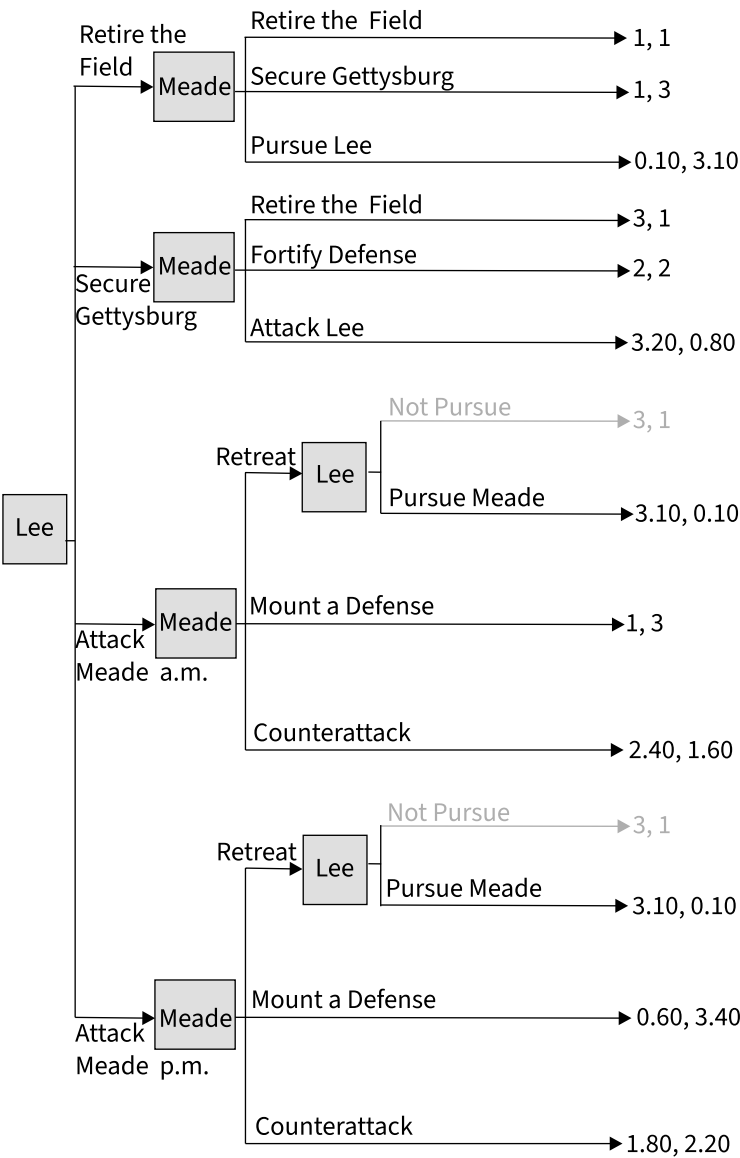


Figure 5.12: *The Battle for Gettysburg*, with General Lee's best response to the sub-games in which General Meade's course-of-action choice is to retreat.

Using the exact same analytical process, Meade's last three decision nodes can be resolved down to a "best" course-of-action choice. Meade's best course-of-action choice, contingent upon Lee securing his position at Gettysburg, is to fortify his current position and prevent farther incursion by Lee. This course-of-action choice yields Meade with his highest contingent-outcome valuation of two. Meade's best course-of-action choice, contingent upon Lee choosing to attack in the morning, is to fight from a defensive position. This course-of-action choice yields Meade with his highest contingent-outcome valuation of three. Meade's best course-of-action choice, contingent upon Lee choosing to attack in the morning, is to fight from a defensive position. This course-of-action choice yields for Meade his highest contingent-outcome valuation of 3.40.

Resolving *The Battle for Gettysburg* up to Meade's best course-of-action-choices, contingent upon Lee's course-of-action choice to initiate the game, as illustrated in Figure 5.13, gives Lee the strategic view that allows him to see his initial course-of-action choice all the way through to its outcome valuation.

Starting with Lee's first course-of-action choice, to retire from the field, Lee can see that Meade's best course-of-action choice is to pursue him. This choice sequence leads to the nonstrategic player Nature resolving the pursuit and Lee's expected outcome valuation is 0.10, as shown in the top course-of-action sequence not grayed out in Figure 5.13.

Lee's second course-of-action choice is to secure a defensive stance in Gettysburg. Lee can see that Meade's best course-of-action choice is to fortify his defensive position and block any farther incursion by Lee. This choice sequence leads to an outcome valuation of two for Lee.

Lee's third course-of-action choice is to attack Meade in the morning. Lee can see that Meade's best course-of-action choice to his morning attack is to fight a defensive battle. This choice sequence leads to an outcome valuation of one for Lee.

Gettysburg: Lee's Sub-Games Resolved

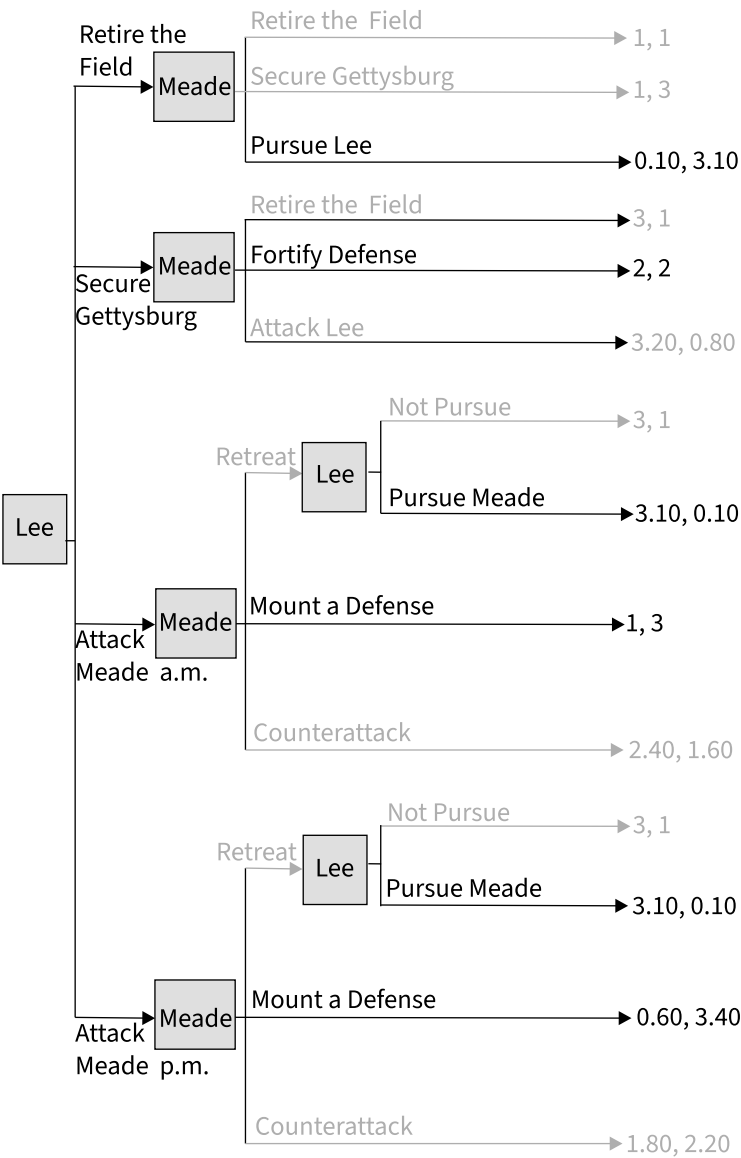


Figure 5.13: *The Battle for Gettysburg*, with all of General Meade's sub-game best responses to General Lee's first-level course-of-action choices.

Lee's fourth course-of-action choice is to attack Meade in the afternoon. Lee can see that Meade's best course-of-action choice to his afternoon attack is to fight a defensive battle. This choice sequence leads to an outcome valuation of 0.60 for Lee.

Meade's best course-of-action choice to Lee's choice to retire from the field yields Lee an outcome valuation of 0.10. Meade's best course-of-action choice to Lee's choice to secure Gettysburg yields Lee an outcome valuation of two. Meade's best course-of-action choice to Lee's choice to commit to a morning attack yields Lee an outcome valuation of one. Meade's best course-of-action choice to Lee's choice to an afternoon attack yields Lee an outcome valuation of 0.60. Looking at his course-of-action choices contingent on their outcome valuations, Lee can determine that, given Meade's best course-of-action choices to Lee's course-of-action choice, Lee's best course of action is to secure Gettysburg and earn a valuation of two, as illustrated in Figure 5.14 with the bolded course-of-action sequence.

The fully specified equilibrium course-of-action choices are as follows: Lee chooses the course of action to secure Gettysburg, and Meade chooses the contingent course of action to fortify defenses. If Lee chooses to retire the field, then Meade chooses to pursue Lee. If Lee chooses to attack Meade in the morning, then Meade chooses to mount a defense.

With Lee and Meade making "best" course-of-action choices contingent on the other's choice-of-action choices, the game resolves to a sub-game perfect equilibrium. This is where neither Lee nor Meade can achieve a higher outcome valuation by a unilateral course-of-action choice change. That is, once at the equilibrium, a change in course-of-action choice by Lee will leave him worse off, or a change of course-of-action choice by Meade will leave him worse off as well. The sub-game perfect equilibrium occurs with first mover Lee choosing to secure Gettysburg. In response, second mover Meade chooses to fortify his defensive position. As a result, Lee holds some Northern territory, holds Gettysburg opposed, with an outcome valuation of two. Meade stands in opposition, blocking any further incursion by Southern forces, with an outcome valuation of two.

Given the simplifying assumptions used to build this game, we would predict that Lee and Meade would proceed as predicted by the equilibrium. But, in the historical record, the game played out quite differently. Lee did not choose to secure and defend his position at Gettysburg. Instead, Lee chose to attack Meade and the game played out poorly for Lee as shown with the dashed course-of-action sequence in Figure 5.15.

Lee had been advised by his top strategist, Longstreet, that a defensive posture was best for the Army of Northern Virginia. Longstreet made two recommendations: to secure Gettysburg and wait for a response from Meade, and to move to defensible ground between Meade and Washington, forcing Meade to attack a defended position. Both suggestions would result in Lee fighting a defensive battle, forcing Meade to be the attacker. Lee decided to attack Meade on the second day.

Here is where the framework developed through game theory could have been useful. Longstreet could have made a compelling argument for the course-of-action choice to secure Gettysburg and force Meade to respond. The outcome would have been the equilibrium illustrated in Figure 5.14. Lee would have owned Northern soil, and Meade would have significant forces tied up in Pennsylvania. A subordinate officer with a compelling strategic understanding of the situation would have a powerful tool to express his strategic vision as something more than an educated opinion.

Lee may have been more open to accepting Longstreet's advice if there was a compelling logical structure supporting the defensive course-of-action choice. This is one of the benefits of game theory to strategic decision-makers. Game theory gives the intuitive strategist a powerful method to *explain* their understanding of a decision environment. In this case, Longstreet would have had a tool that could have clearly and directly conveyed the breadth of possible scenarios available in the game to Lee and which scenario would play out to the South's best advantage.

Gettysburg: The Historical Outcome

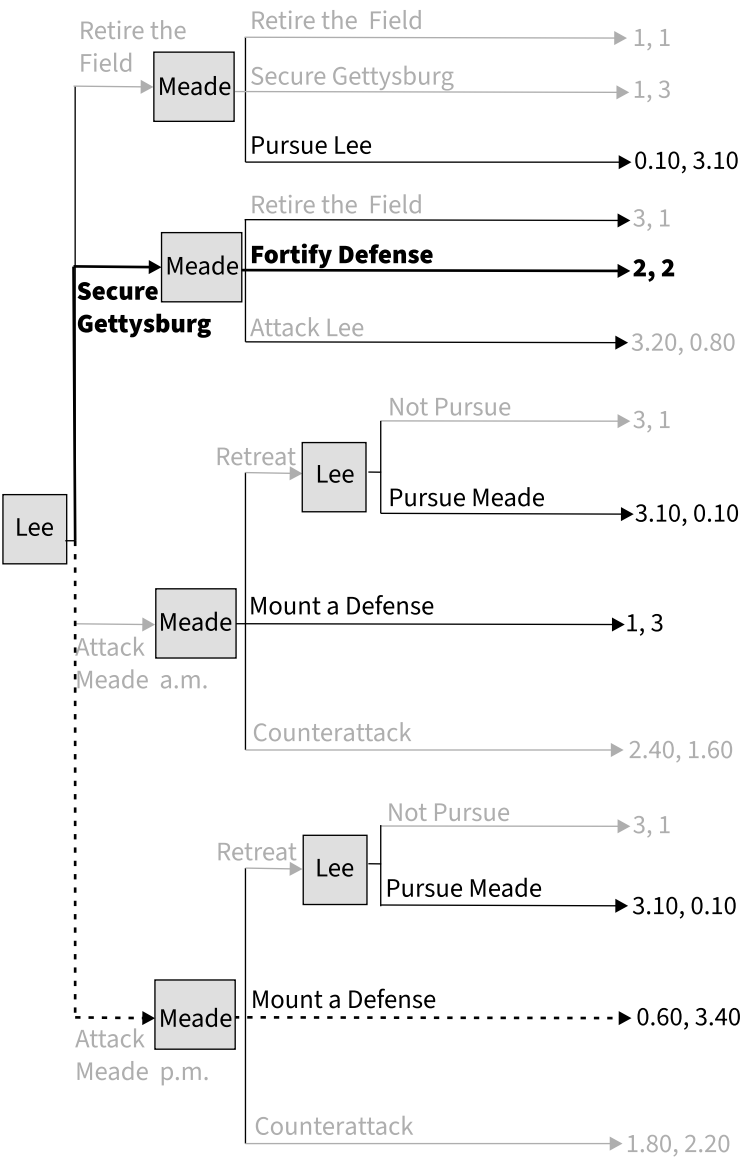


Figure 5.15: *The Battle for Gettysburg* played to a suboptimal outcome for General Lee. He attacked General Meade, who defended his position, yielding an outcome valuation of 0.60 for Lee and 3.40 for Meade.

In the next three chapters, the role of information and disinformation in a strategic-choice environment will be added. Up to this point, the players of the game had identical and mostly complete information. The only uncertainty was that introduced by the nonstrategic and stochastic player, Nature. But even here, the players both knew and were fully aware of the probability distribution across which Nature resolved its choice.

CHAPTER VI

A FEINT WITHIN A FEINT

*Warfare is the Way [Tao] of deception.
Thus although [you are] capable, display
incapability. When committed to employing your
forces, feign inactivity. When [your objective]
is nearby, make it appear distant; when far
away, create the illusion of being nearby.*

SUN TZU, *ART OF WAR*

Up to this point, the only way we have discussed uncertainty and applied it into our modeling has been through the random nonstrategic player called Nature. Uncertainty can also be introduced into a game through differing levels of understanding about the game environment between the players. This type of uncertainty is called “information asymmetry.”¹ Another way uncertainty can be introduced (or resolved) into a game is through the direct actions of the players. These actions can send a signal (information or disinformation) as to the nature of the game.^{2, 3}

The hockey scenario in the introduction to this book illustrates how one player can intentionally introduce uncertainty into a strategic interaction. In that example, the center had control of the puck at center ice and looked with his head to the left but passed the puck to his right wing. The center’s ambiguous look resulted in one defenseman out of position.

The interaction between the center and the defenseman has two different implications concerning the introduction of uncertainty by a strategic player. The first implication in this passage is that the defenseman cannot know whether the action taken by the center is intends to pass the puck to his right wing or to hold onto the puck himself and carry it to his left until after the defenseman has chosen his course of action. In the above example, the center has already chosen to pass the puck to his wing on the right, but the defenseman cannot know the center’s choice until he has committed to his defensive course of action. This is the type of uncertainty that is in the fundamental structure of a simultaneous-play game and is not disinformation. The center and defense choose courses of action without the prior knowledge of their opponents. But in this case, both players are at similar advantage. They both have the same level of information in the game.

The second form of uncertainty introduced in the hockey example is that the center can send a signal, an action that contains information or disinformation regarding his intentions to the other players. In the passage above, the center head faked to the left. The defenseman interpreted this to mean that the center was going to carry the puck to the left and acted upon that disinformation. The center knew what course of action he was going to carry out, but the defense did not know and was at an informational disadvantage to the center.

The first form of uncertainty is the fundamental strategic uncertainty innate to the structure of a simultaneous-play game. That is, no player “knows” the course-of-action choice of an opposing player until the game resolves itself to the outcome stage. The second and more interesting form of uncertainty comes from differences in information about the game between the players and this type of strategic uncertainty that can occur in both simultaneous-play and sequential-play games. If this asymmetric informational uncertainty exists in a sequential-play game, the following player must choose a course of action without knowing exactly what the preceding player has chosen.

Sequential-Play Games with Uncertainty

To understand a sequential-play game with informational asymmetry and its corresponding uncertainty, let us consider a twist to the *Sequential-play Game of Siege* introduced in Figure 3.1. Let us suppose the city has the additional course of action to ambush the attacking raiders. That is, the city can choose to close up the city, set an ambush for the raiders, or fully engage the raiders with a direct attack. In addition, assume there is no way for the raiders to tell whether the city is closed or whether the city has set up an ambush. This observational equivalence is the source of the information asymmetry: The city knows whether it has set an ambush or has closed up, but the raiders do not.

Because the city has three course-of-action choices (to close, to ambush, and to attack) and the raiders have two course-of-action choices (to siege and to avoid), there are six possible outcomes to the game. These six outcomes are laid out in Table 6.1. The outcomes are preferred by the city such that (No Casualties, No Ransom) is preferred to (Few Casualties, No Ransom) is preferred to (Some Casualties, No Ransom) is preferred to (No Casualties, Ransom Paid) is preferred to (Many Casualties, Some Looting) is preferred to (Many Casualties, City Looted), and their preferences are valued from one, for least preferred, to six, the most preferred. The raiders' outcome valuations are listed in the fifth column.

Table 6.1: *Game of Siege with Uncertainty Outcomes*

The course-of-action choice pairs for the game, their result and outcomes, and preference valuations are listed for the city and the raiders in the table below.

CITY COA	RAIDERS' COA	OUTCOME	CITY VALUE	RAIDERS' VALUE
Close	Siege	No Casualties, Ransom Paid	3	6
Close	Avoid	No Casualties, No Ransom	6	3
Ambush	Siege	Many Casualties, Some Looting	2	4
Ambush	Avoid	Few Casualties, No Ransom	5	1
Attack	Siege	Many Casualties, City Looted	1	5
Attack	Avoid	Some Casualties, No Ransom	4	2

This new game is represented in Figure 6.1. The city’s leaders know whether they have agreed to close up, set an ambush, or to attack. This is represented by the three distinct course-of-action rays emanating from the city’s decision node. The raiders can directly observe whether the city has chosen to attack. This information is indicated, with certainty, by the attack ray leading directly to the raiders’ isolated decision node, the city attacks. The raiders can know that the city has chosen to attack and can respond directly to the attack.

The raiders, on the other hand, cannot tell the difference between an ambush and a closed-up city. To ambush and to close are observationally equivalent courses of action to the raiders. The city knows whether it has closed or set an ambush, but the raiders cannot tell. To indicate this information asymmetry, we draw an oval around the raiders’ two decision nodes associated with closing and ambushing. To the raiders, the information available leaves them with two information sets. The first information set is the decision node associated with the city leaders choosing to attack. The second information set is when the raiders observe that the city gates are closed with no observable force facing them. Has the city closed itself in, or has it set an ambush?

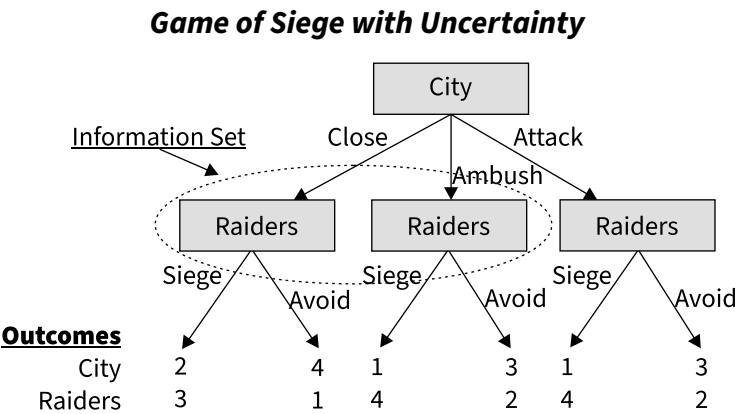


Figure 6.1: The raiders are unable to tell whether the city has closed its gates or set an ambush until the raiders commit to a siege or to avoid the city. The dotted oval around the raiders’ decision nodes at close and ambush represents their information set.

The forward-looking process of backward induction can only strictly be applied to Sub-game 1 in this example. In Sub-game 1, the raiders can choose between laying siege for an outcome value of four and avoid for an outcome value of two. Given that the city has chosen to attack, and the raiders know they are at their decision node in Sub-game 1, the raiders will clearly choose to lay siege for an outcome of four, as indicated in Sub-game 1 in Figure 6.3.

In Sub-game 2, the raiders do not know at which of the two possible nodes they are located. But, as the game is configured, the raiders know they will either gain an outcome value of six or four if they choose to lay siege or an outcome value of three or one if they choose to avoid the city. Comparing the outcome pairs, the raiders notice how—regardless of the decision node where they actually are located—choosing to lay siege always gives them a better outcome if the city chooses to close, or if the city chooses to ambush, rather than choosing to avoid the city, as indicated in Sub-game 2 of Figure 6.3.

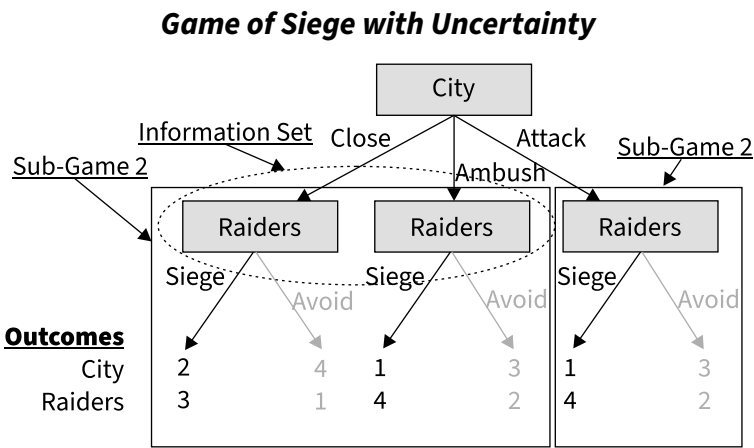


Figure 6.3: Sub-game 1 easily resolves up one level, as described in the discussion in Chapter III. In Sub-game 2, the raiders must compare their outcomes, across their strategy choices and choose the strategy that is acceptable across both of its potential nodes.

In this game, the raiders have a Dominant Course of Action to lay siege regardless of the course of action chosen by the city. Resolving the *Game of Siege with Uncertainty* up one level, the city is left to choose among the choice to close with an outcome value of three, ambush with an outcome value of two, and attack with an outcome value of one. Given this set of choices, the city will choose to close, the raiders will choose to lay siege, and the game will have a sequential equilibrium outcome value of three to the city and six to the raiders, as indicated in Figure 6.4.

To resolve Sub-game 2 up to the next level, the raiders are required to choose a strategy that is *satisfactory* across both possible nodes within the information set.⁴ If the siege game had different outcomes and the raiders did not have the unambiguous course-of-action choice to siege, then how would they determine which course of action to choose if they found themselves in Sub-game 2?

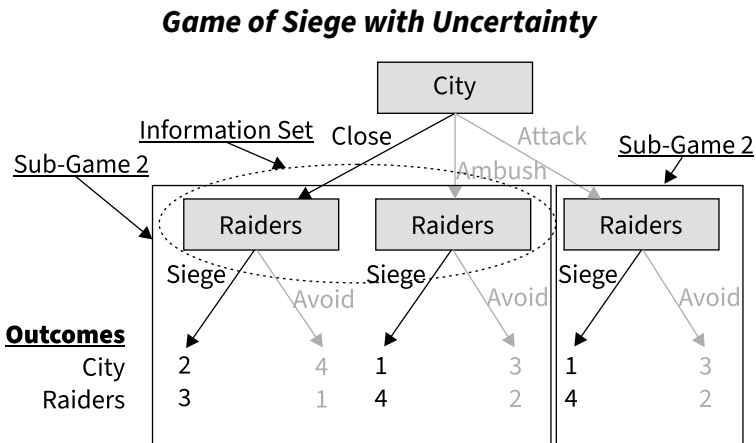


Figure 6.4: The Raiders have a dominant strategy to siege. The city faces outcomes of three if they choose to close, one if they choose to ambush, and one if they choose to attack. The city will clearly choose to close, and the raiders will choose to siege, achieving the sequential equilibrium outcome of two to the city and three to the raiders.

Mixing Simultaneous-Play and Sequential-Play Games Another way to visualize the *Game of Siege with Uncertainty* presented in Figure 6.1 is to recognize that, although the game plays out sequentially, the information set for the raiders in Sub-game 2, and the information held by the city, has close similarities with a simultaneously played game. That is, though the city chooses and acts on whether to close up or whether to ambush before the raiders choose their course of action, the raiders cannot discern the city leaders' course-of-action choice until they have committed to their own and the game resolves to the payoff stage. Sub-game 2 plays out as if both the city and the raiders choose their courses of action simultaneously. Sub-game 2 of the *Game of Siege with Uncertainty* is highlighted in Figure 6.5.

Sub-game 2, in which the city chooses between the two courses of action to close and to ambush, and the raiders choose between the courses of action to siege and to avoid, can be reimagined as the

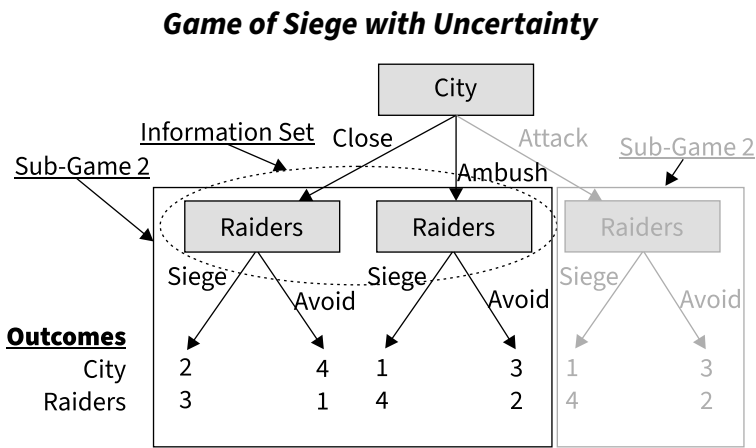


Figure 6.5: Looking at only Sub-game 2, the information sets held by both the city and the raiders are similar to the information sets they would have if that part of the game were played simultaneously instead of sequentially.

simultaneous-play game illustrated in Figure 6.6. The city leaders do not know the raiders' choice because in the sequential play of the game, they have chosen before the raiders. The raiders do not know the choice of the city leaders, even though the raiders choose after the city leaders have chosen, because the city's choice to ambush or to close is observationally equivalent to the raiders'.

Thus, portraying Sub-game 2 as a simultaneous-play game, the *Game of Siege with Uncertainty* can also be portrayed as a compound game, with a portion that is sequential and a portion that is simultaneous. This hybrid game is represented in Figure 6.7. If the city leaders initiate game play by choosing to attack the raiders, the players find themselves in Sub-game 1, and it is analyzed as any sub-game in the standard sequential-play game format. If the city leaders choose to ambush the raiders or close the gates, then the players find themselves in Sub-game 2 and it is analyzed in the standard simultaneous-play game format.

Game of Siege with Uncertainty

		Raiders	
		Siege	Avoid
City	Close-up	3 6	6 3
	Ambush	2 4	5 1

Figure 6.6: Because the information set held by the raiders does not allow them to know whether the city has closed or laid an ambush, Sub-game 2 can be represented as a simultaneous-play game.

Game of Siege with Uncertainty

Sub-Game 2

Nash Equilibrium

		Raiders	
		Siege	Avoid
City	Close-up	3 ↑	6 ↑ 3 ←
	Ambush	2 ↓	5 ↓ 4 ← 1

Figure 6.8: Using the mutual best-response method to find Sub-game 2's Nash equilibrium, the course-of-action pair (close, siege) yields the Nash equilibrium outcome valued at three for the city and six for the raiders.

Sub-game 2, the Nash equilibrium is that the city chooses the course of action to close its gates and the raiders choose the course of action to lay siege, yielding an outcome valuation of three for the city and an outcome valuation of six for the raiders. Given this set of choices and outcome valuations, the city will play Sub-game 2 by choosing the course of action to close and the raiders will choose the course of action to lay siege. These course of action choices will yield an equilibrium in which the city has an outcome valuation of three and the raiders have an outcome valuation of six, as illustrated in Figure 6.9.

The *Game of Siege with Uncertainty* was designed with dominant strategies so that intuition would support the game-theoretic solution techniques. It is not common that the game played has a mutual dominant-strategy solution so that intuition may fail and the game solution techniques must be relied upon.

Solving the *Game of Siege with No Dominant Strategies* To escape the world of mutual dominant strategy equilibria, consider the outcome valuations for the *Game of Siege with Uncertainty* presented in Table 6.2 and the new game presented in Figure 6.10.

Game of Siege with Uncertainty

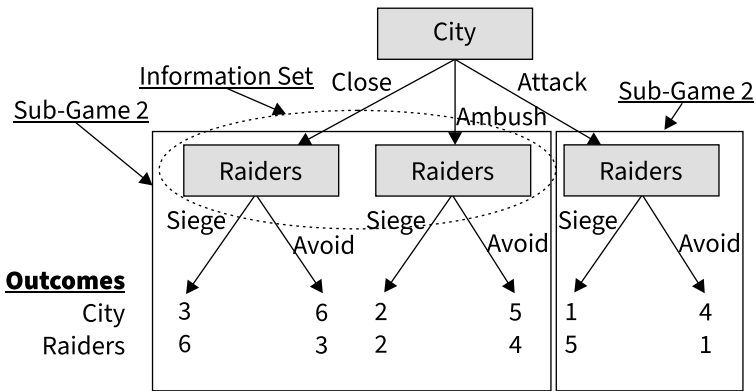


Figure 6.10: In this version of the *Game of Siege with Uncertainty*, the raiders no longer have a dominant strategy to siege in Sub-game 2. If the city chooses the course of action to close, the raiders choose the course of action to siege with an outcome valuation of six. If the city chooses the course of action to ambush, the raiders choose the course of action to avoid, with an outcome valuation of four.

In this new representation of the siege game, the raiders no longer have an unambiguous course-of-action choice to siege in Sub-game 2. Depending upon the decision node the raiders find themselves at in the information set, they would prefer to engage in different course of action choices.

In Sub-game 1, the raiders, finding themselves attacked by the city, would choose the course of action to siege to get an outcome valuation of five, which is greater than the outcome valuation of one if they chose the course of action to avoid. The city, looking forward to the raiders' rational response to the course-of-action choice to attack, knows that playing in Sub-game 1 would give it an outcome valuation of one. All the city's outcome valuations in Sub-game 2 exceed one. If the raiders assumed a rational city, they would expect to find themselves playing in Sub-game 2 of the *Game of Siege with Uncertainty*.

The information set in Sub-game 2 from Figure 6.10 can be represented as the simultaneous-play game illustrated in Figure 6.11. Starting in the bottom left cell of the simultaneous-play representation of Sub-game 2 in Figure 6.12, if the city chooses to ambush and the raiders choose to siege, the city can benefit from a unilateral course-of-action choice change from ambushing the raiders to closing its walls, which would increase their outcome valuation from two to three. Similarly, the raiders can benefit from a unilateral course-of-action choice change from laying siege to avoiding the city, which would increase their outcome valuation from two to four. Because neither the city nor the raiders find their course of action choices to be their best individual responses to the other's course of action choices, that cannot be a Nash equilibrium. The only cell in which each course-of-action choice is a best response to the other's choice is the top left cell. The Nash equilibrium to Sub-game 2 is for the city to choose to close its gates and for the raiders to choose to lay siege, with an outcome valuation of three to the city and an outcome valuation of six to the raiders, as indicated in Figure 6.12.

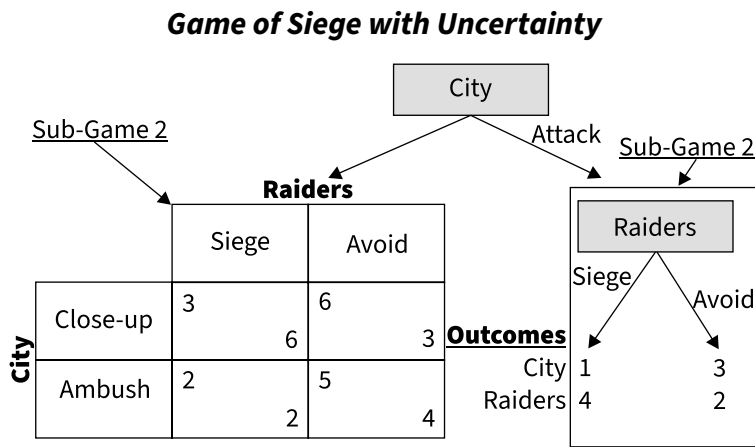


Figure 6.11: Substituting the simultaneous-play equivalent game for the sequential-play information set in Sub-game 2 gives the above compound game.

Game of Siege with Uncertainty

Sub-Game 2

Nash Equilibrium

		Raiders	
		Siege	Avoid
City	Close-up	3 ↑	6 ↑ 3
	Ambush	2 ↓	5 ↓ 4 ← 1

Figure 6.12: Using the mutual best-response method to find Sub-game 2's Nash equilibrium, the course-of-action pair (close, siege) yields the Nash equilibrium outcome valued at three for the city and six for the raiders.

Reassembling the sequential game with its various sub-games resolved up to the city's course-of-action choice yields Figure 6.13. The city can choose to attack and play Sub-game 1, yielding it an outcome valuation of one, or the city can choose to play Sub-game 2. The Nash equilibrium to the simultaneous-play game in Sub-game 2 is the city chooses the course of action to close and the raiders choose the course of action to siege, with their respective outcome valuations of three for the city and six for the raiders. Since the city's outcome valuation for playing in Sub-game 2 is higher than its outcome valuation for playing in Sub-game 1, the city will choose to play Sub-game 2 and will choose the course of action to close. The raiders, finding themselves playing Sub-game 2, will choose the course of action to lay siege. This yields the sub-game perfect Nash equilibrium to the *Game of Siege with Uncertainty* in which the city chooses to close for an outcome valuation of three and the raiders choose to lay siege with an outcome valuation of six.

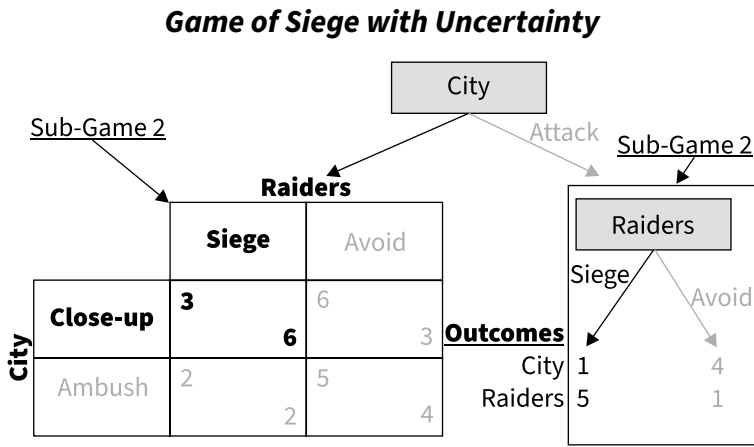


Figure 6.13: Given the Nash equilibrium of Sub-game 2 and the sequential strategy of the raiders in Sub-game 1, the city has the choice of playing Sub-game 1 by choosing to attack or playing Sub-game 2 by choosing to close or to ambush. The city, when choosing between attack and an outcome valuation of one in Sub-game 1, and close with an outcome valuation of three in Sub-game 2, will choose to close. This yields the sub-game perfect Nash equilibrium to the *Game of Siege with Uncertainty*, in which the city chooses to close for an outcome valuation of three and the raiders choose to siege with an outcome valuation of six.

Figure 6.14 gives the original sequential-play structure with information sets of the *Game of Siege with Uncertainty* as illustrated in Figure 6.10. The incentives faced by the city and the raiders lead to the city's and the raiders' course of action choices in which the city chooses to close and the raiders choose to siege, yielding the outcome valuations of three for the city and six for the raiders.

The Siege Game with Multiple Equilibria Consider the siege game in which the course of action outcome valuations for the city and the raiders are reconfigured as presented in Table 6.3. In this new version of the *Game of Siege with Uncertainty*, the city's preference valuations for the outcomes (No Casualties, No Ransom) and (Few Casualties,

Game of Siege with Uncertainty

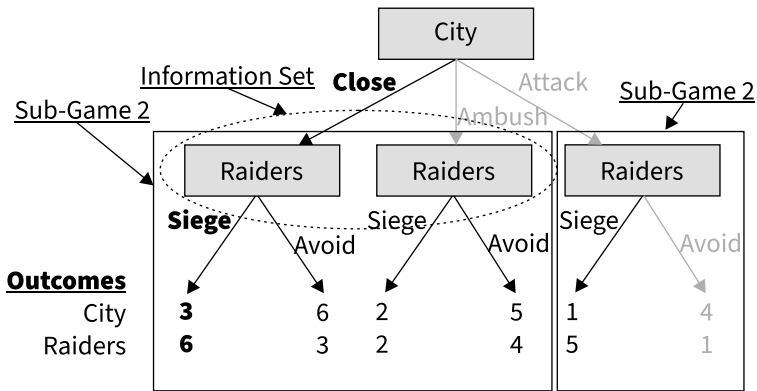


Figure 6.14: In this version of the *Game of Siege with Uncertainty*, the city's and the raiders' course-of-action choices lead to the equilibrium in which the city chooses to close and the raiders choose to siege, yielding the outcome valuations of three for the city and six for the raiders.

No Ransom) are switched. The city now has a preference valuation of five for the outcome (No Casualties, No Ransom) and a preference valuation of six for the outcome (Few Casualties, No Ransom). It is important to note that in the previous version of the *Game of Siege with Uncertainty*, the city had a dominating strategy in Sub-game 2 to choose the course of action to close. Under this new configuration of the game, the city no longer has the dominating course-of-action choice to close up in Sub-game 2. In fact, in this configuration of the game, neither the city nor the raiders have dominant strategies in Sub-game 2.

The new formulation of the *Game of Siege with Uncertainty*, using the outcome valuations as presented in Table 6.3 is illustrated in Figure 6.15.

Table 6.3: *Game of Siege with Uncertainty Outcomes*

The course-of-action choice pairs for the game, their result and outcomes, and preference valuations are listed for the city and the raiders in the table below.

CITY'S COA	RAIDERS' COA	OUTCOME	CITY'S VALUE	RAIDERS' VALUE
Close	Siege	No Casualties, Ransom Paid	3	6
Close	Avoid	No Casualties, No Ransom	5	4
Ambush	Siege	Many Casualties, Some Looting	2	2
Ambush	Avoid	Few Casualties, No Ransom	6	3
Attack	Siege	Many Casualties, City Looted	1	5
Attack	Avoid	Some Casualties, No Ransom	4	1

Game of Siege with Uncertainty

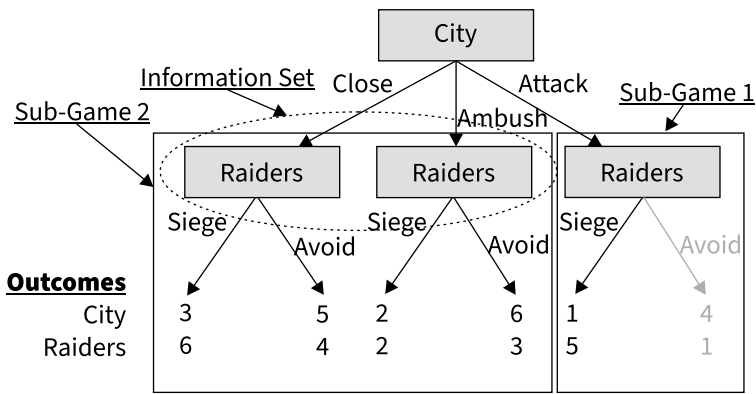


Figure 6.15: In this new configuration of the *Game of Siege with Uncertainty*, neither the city nor the raiders have dominant strategies in Sub-game 2.

As nothing has changed in Sub-game 1 of the *Game of Siege with Uncertainty*, the logic of evaluating the raiders' response to the city's course-of-action choice to attack, when the raiders can know an attack is forthcoming, is to lay siege. Thus, if the city and the raiders find themselves in Sub-game 1, the city will achieve an outcome valuation of one and the raiders will achieve an outcome valuation of five, as in all the previous versions of the game, as indicated in Sub-game 1 of Figure 6.15.

To move forward in solving this game, Sub-game 2 can be transformed into the simultaneous-play game as represented in Figure 6.16. Using the mutual best-response method for finding Nash equilibria, as demonstrated by the arrows in Figure 6.16, two distinct equilibria emerge in Sub-game 2: the course of action pair (close, siege) yielding the Nash equilibrium outcome valued at three for the city and six for the raiders and the course of action pair (ambush, avoid) yielding the Nash equilibrium outcome valued at six for the city and four for the raiders. Sub-game 2 of the *Game of Siege with Uncertainty* now has two equilibria instead of one.

Game of Siege with Uncertainty

Sub-Game 2

Nash Equilibria

		Raiders	
		Siege	Avoid
City	Close-up	3, 6	5, 4
	Ambush	2, 2	6, 3

Figure 6.16: Using the mutual best-response method to find Sub-game 2's Nash equilibria, the course-of-action pair (close, siege) yields the Nash equilibrium outcome valued at three for the city and six for the raiders, and the course-of-action pair (ambush, avoid) yields the Nash equilibrium outcome valued at six for the city and three for the raiders. There is no longer a unique Nash equilibrium for Sub-game 2.

The city's course-of-action choice to attack that initiates game play in Sub-game 1 yields an outcome valuation of one to the city because the raiders will choose the course of action to lay siege. The city's course-of-action choice to attack, in Sub-game 1, clearly yields lower outcome valuations to the city than either Nash equilibrium in Sub-game 2. The Nash equilibrium in Sub-game 2 characterized by the city choosing the course of action to close and the raiders choosing the course of action to siege yields an outcome valuation of $3 > 1$ to the city. The Nash equilibrium in Sub-game 2, characterized by the city choosing the course of action to ambush the raiders, and the raiders in turn choosing the course of action to avoid, yields an outcome valuation of $6 > 1$ to the city. With this knowledge the city will not choose the course of action to attack, and Sub-game 1 can be eliminated from being the source of a potential equilibrium to the *Game of Siege with Uncertainty*, as shown in Figure 6.17.

Having eliminated the city's course-of-action choice attack from the city's rational game play and Sub-game 1 from consideration for equilibrium game play, the city and the raiders find themselves playing Sub-game 2. Since Sub-game 2 does not have a unique Nash equilibrium, neither the city nor the raiders have distinct course of action choices using the backward induction solution process. Neither can look forward and reason back to know what course-of-action choice the other will make. In its sequential-play format, as represented in Figure 6.18, the two Nash equilibria for the *Game of Siege with Uncertainty* fall inside Sub-game 2 and are also inside the raiders' information set. The raiders cannot know which equilibrium will emerge, as they cannot know the disposition of the city's forces.

This is not a particularly satisfying equilibrium outcome to the *Game of Siege with Uncertainty*. All that a strategist for the city has to offer is, do not select the course of action attack because if we do, the raiders will choose the course of action siege and that would be bad for us. All a strategist for the raiders has to offer is, if the city chooses the course of action attack, we will choose the course of action siege; the city knows this, so it will not select the course of action attack. The city will either choose the course of action to close or to ambush.

Game of Siege with Uncertainty

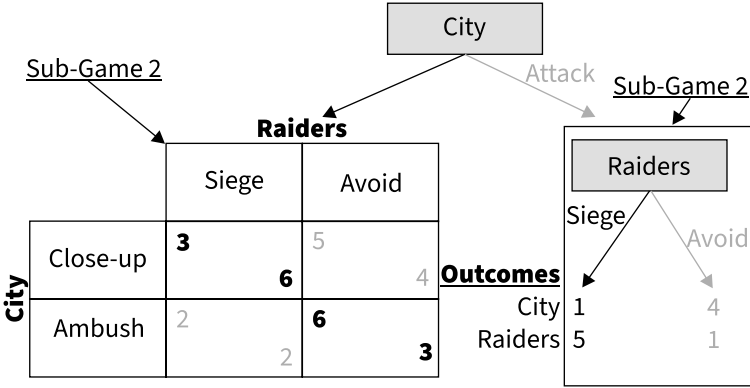


Figure 6.17: Given Sub-game 2 does not have a unique Nash equilibrium, neither the city nor the raiders have distinct course-of-action choices using the backward induction solution process. Neither can know what course-of-action choice the other will make.

Game of Siege with Uncertainty

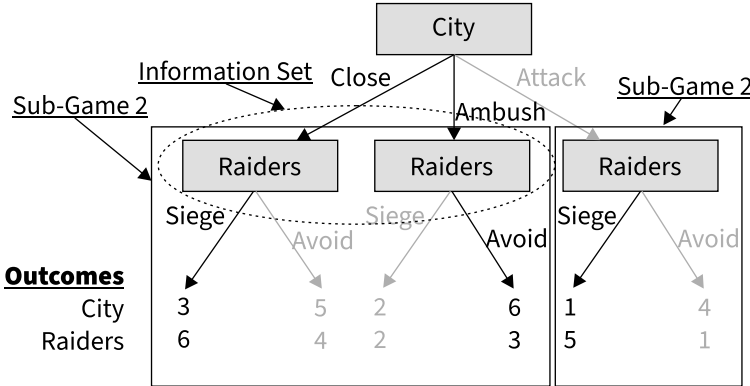


Figure 6.18: Reconfiguring the *Game of Siege with Uncertainty* back into its sequential-play format, the two Nash equilibria fall inside Sub-game 2 and are also inside the raiders' information set. The raiders cannot know which equilibrium will emerge as they cannot know the disposition of the city's forces.

To resolve this outcome away from the unknowable, the raiders need a deeper understanding of how the city leaders tend to choose courses of action that reside inside of the raiders’ information set. If the raiders could perfectly distinguish when the city chooses the course of action to close and when the city chooses the course of action to ambush, the *Game of Siege with Uncertainty* would have no uncertainty and would resolve as the games discussed in Chapter III and as presented in Figure 6.19.

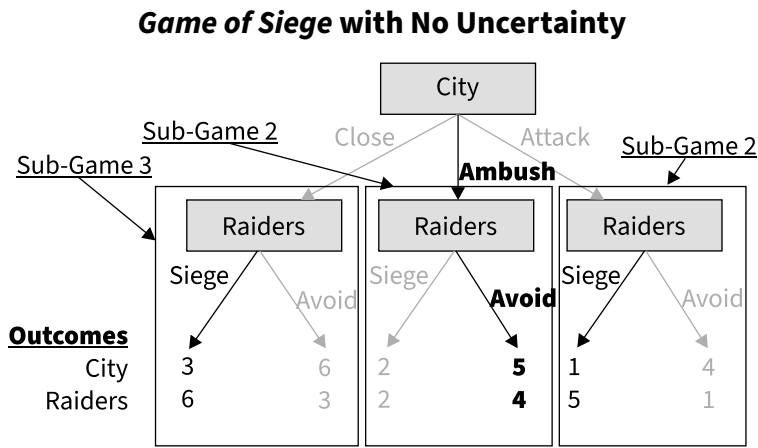


Figure 6.19: Starting from the raiders’ course-of-action choice set, the raiders will choose to siege if in Sub-game 1, they will choose to avoid if in Sub-game 2 and will choose to siege if in Sub-game 3. The city, knowing the raiders’ course-of-action choice set will choose the course of action to ambush, yielding the sub-game perfect Nash equilibrium in which the city chooses the course of action to ambush and the raiders choose the course of action to avoid, yielding an outcome valuation of six to the city and an outcome valuation of three to the raiders.

In most instances in such games with uncertainty, the most a player can know about another player is the probability distribution across the other player’s course-of-action choice set. The raiders may not be able to discern whether the city leaders have chosen the course of action to ambush nor whether the city has chosen the course of action

to close when in Sub-game 2, but they may know the frequencies at which the city chooses one course of action or the other. In Chapter VII, we will add information about a player's probability distribution to the *Game of Siege with Uncertainty* and develop the techniques to resolve the game to equilibrium.

CHAPTER VII

**EXPECTATIONS
ABOUT A
CUNNING OPPONENT**

*For this reason the wise contemplate the
intermixture of gain and loss. If they discern
advantage [in difficult situations], their efforts
can be trusted. If they discern harm [in prospective
advantage], difficulties can be resolved.*

SUN TZU, ART OF WAR

As seen in Chapter VI, players who face an information set in which multiple decision nodes are observationally equivalent may be left in a position where they have no good method for choosing an optimal course of action. In the case of the *Game of Siege with Uncertainty* last presented, the city could choose the course of action to close its gates or the course of action to ambush. These two course-of-action choices were indistinguishable by the raiders creating an information set for the raiders where they did not know the city's course-of-action choice. Under certain conditions in which particular courses of action were dominant, the game could be resolved to a sub-game perfect Nash equilibrium, but those circumstances were quite unlikely to occur in a natural setting.

In this chapter, the information requirements, short of perfect foresight on the part of the raiders, to resolve the *Game of Siege with Uncertainty* will be introduced. Though the raiders do not know the course-of-action choice made by the city inside of a multi-node information set, they may know, or have an idea about, the frequencies at which the city chooses one course of action or another. Thus, the raiders have a probability distribution across their decision nodes inside an information set.

There are two different ways to think about how these probabilities come into existence. The first is to consider that there are two city types. There are aggressive cities that always lay ambushes given they do not choose to attack, and there are passive cities that always close their gates given they do not choose to attack. The city type is not observationally obvious to the raiders, but the proportion of the cities that are one type or the other is known to the raiders.

The second is to consider that the city leaders assign a probability distribution across their available courses of action. That is, the city

randomly chooses to ambush with a particular probability and to close with another probability to keep the raiders on their toes. The raiders do not know which course of action the city may choose, but they do have beliefs about the probability distribution.

Bayesian Nash Equilibrium

The first way we discussed adding uncertainty into game play was to consider that there were two city types. There were aggressive cities that have the course-of-action choices to ambush and to attack, and there were passive cities that have course-of-action choices to close their gates and to attack. As in the previous game, the raiders do know if either city type has chosen the course of action to attack. But since the city type is not observationally obvious to the raiders, the raiders do not know which type of city they are approaching, and the raiders cannot distinguish the course-of-action choice to ambush from the course-of-action choice to close the gates. The raiders do know the proportion of the cities that are aggressive or passive. The course-of-action choice pairs, their associated outcomes, and the outcome valuations for the raiders and both city types are presented in Table 7.1.

In the game described in Table 7.1, the raiders come upon a city not knowing the city type but knowing the probability distribution of city types as illustrated in Figure 7.1. The raiders know and can observe if a city's leadership is choosing the course of action to attack, regardless of city type. But the raiders cannot tell if a city has chosen the course of action to close or to ambush. The raiders' inability to distinguish between an aggressive city choosing the course of action to ambush and a passive city choosing the course of action to close its gates is represented by the raiders' information set that straddles the two city types in Figure 7.1.

Table 7.1: *Game of Siege* with Uncertain City Types

The course-of-action choice pairs for the *Game of Siege* with uncertain city types, coupled with their outcomes and their preference valuations given two city types.

Game When the City Is of Type Passive				
CITY'S COA	RAIDERS' COA	OUTCOME	CITY'S VALUE	RAIDERS' VALUE
Close	Siege	No Casualties, Ransom Paid	3	6
Close	Avoid	No Casualties, No Ransom	5	4
Attack	Siege	Many Casualties, City Looted	1	5
Attack	Avoid	Some Casualties, No Ransom	4	1

Game When the City Is of Type Aggressive				
CITY'S COA	RAIDERS' COA	OUTCOME	CITY'S VALUE	RAIDERS' VALUE
Ambush	Siege	Many Casualties Some Looting	2	2
Ambush	Avoid	Few Casualties, No Ransom	6	3
Attack	Siege	Many Casualties, City Looted	1	5
Attack	Avoid	Some Casualties, No Ransom	4	1

Game of Siege with Uncertain City Types

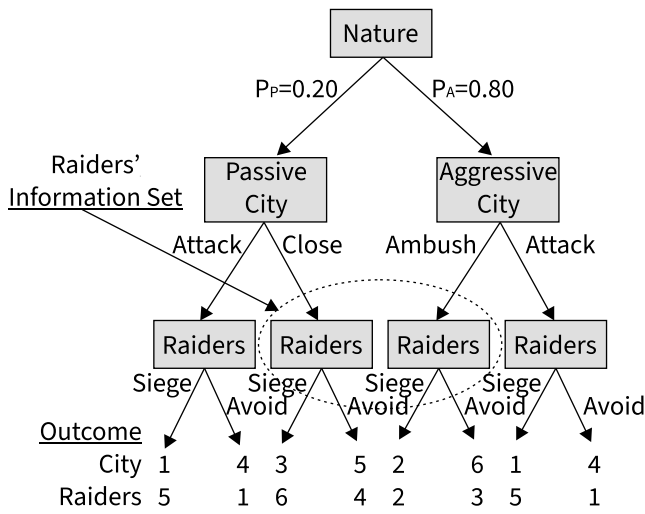


Figure 7.1: There are two city types: aggressive cities that lay ambush or attack and passive cities that close or attack. The city knows its type, but the city type is not observationally obvious to the raiders. The raiders know that 20 percent ($p_{\text{Passive}} = 0.20$) of the cities are passive and 80 percent ($p_{\text{Aggressive}} = 0.80$) of the cities are aggressive.

Nature has the first move in this game as it assigns the city its type. A city is aggressive 80 percent of the time ($P_{\text{Aggressive}} = 0.80$), and a city is passive 20 percent of the time ($P_{\text{Passive}} = 0.20$). Only after game play is engaged is the uncertainty resolved. Once the city type is assigned, the raiders find themselves playing one of two games. If the raiders are not attacked and the city is aggressive, then the city has set up an ambush. If the raiders are not attacked and the city remains passive, then the city has closed its gates.

Using the process of looking forward and reasoning back to find the equilibrium in this game, the raiders find they have three sub-games before them. Sub-game 1 and Sub-game 3 are ones in which they are attacked by the city’s guard, and Sub-game 2 is one in which there are two observationally equivalent decision nodes, as shown in Figure 7.2. Sub-game 1 and Sub-game 3 resolve up a level, with the

raiders choosing their course of action to lay siege in both sub-games. In doing so, the raiders achieve an outcome valuation of five, which exceeds the outcome valuation of one that they would achieve by choosing their course of action to avoid. If they find themselves in the situation where the city, of either type, has chosen its course of action to attack, the raiders will choose to siege. Knowing the course of action to lay siege will be chosen by the raiders, both city types know that if their course-of-action choice is to attack, the game will resolve with them receiving an outcome valuation of one, as demonstrated in Figure 7.2.

Game of Siege with Uncertain City Types

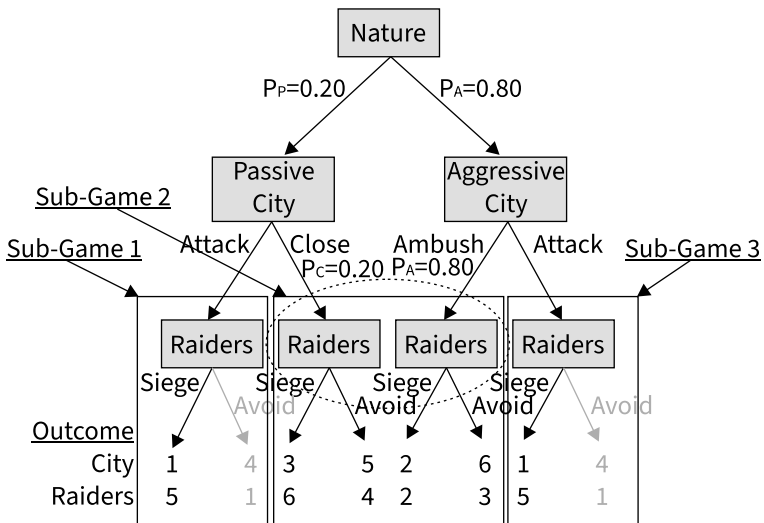


Figure 7.2: The raiders find themselves in one of three sub-games. In Sub-game 2, the conditional probability the raiders are facing a passive city that has closed given the raiders were not attacked is $p_c = 0.20$ and the conditional probability the raiders are facing an aggressive city that has laid ambush given the raiders were not attacked is $p_a = 0.80$.

To resolve Sub-game 2, the raiders need to know the probability distribution across the city's course-of-action choices to close and to ambush—given the city has not yet attacked. Because they know the distribution of city types ($P_{\text{Passive}} = 0.20$ and $P_{\text{Aggressive}} = 0.80$), they can calculate that (given the city has not attacked) they will face a passive city that has chosen the course of action to close, with probability $P_{\text{Close}} = 0.20$, and they will face an aggressive city that has chosen the course of action to ambush, with probability $P_{\text{Ambush}} = 0.80$. With these conditional probabilities, the raiders can now calculate the expected outcome valuation for each of their two course-of-action choices, to avoid and to siege, when they find themselves in Sub-game 2.

The raiders calculate the expected value of the outcome by using the same process described when resolving the outcomes generated by the player, Nature, in Chapter III:

$$\text{Expected Value} = \frac{\text{Probability Outcome 1} \times \text{Value Outcome 1}}{\text{+ Probability Outcome 2} \times \text{Value Outcome 2}}.$$

If the raiders choose as their course of action to avoid, they will face a city that has chosen the course of action to close its gates, with $P_{\text{Close}} = 0.20$, and earn an outcome valued at four, and they will face a city that has chosen the course of action to ambush, with $P_{\text{Ambush}} = 0.80$, and earn an outcome valued at three. This will earn the raiders an expected outcome valuation of $(0.20 \times 4) + (0.80 \times 3) = 3.2$ for choosing the course of action to avoid. Likewise, if the raiders choose to lay siege, they will face a city that has chosen to close, with $P_{\text{Close}} = 0.20$, and earn an outcome valued at six, and they will face a city that has chosen to ambush, with $P_{\text{Ambush}} = 0.80$, and earn an outcome valued at two. This will earn the raiders an expected outcome valuation of $(0.20 \times 6) + (0.80 \times 2) = 2.8$ for choosing the course of action to lay siege, as represented in Figure 7.3.

Game of Siege with Uncertain City Types

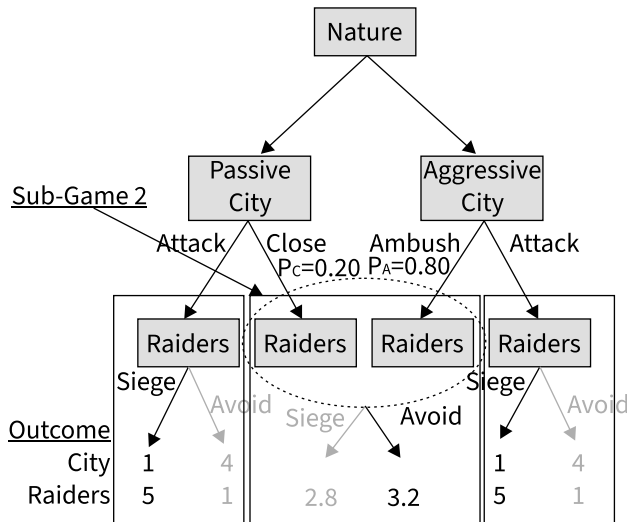


Figure 7.3: In the raiders' information set in Sub-game 2, the raiders' expected outcome values are 2.8 if they choose the course of action to siege and 3.2 if they choose the course of action to avoid. Thus, the raiders will choose to avoid if they are not attacked and will choose to lay siege if they are attacked.

Moving the game up to the decision nodes of the two cities, both city types know that if they choose to attack, the raiders will respond by laying siege, yielding an outcome valuation of one. In Sub-game 2, both cities also know that the raiders' information set leads them to choose their course of action to avoid, achieving the higher expected outcome valuation of 3.2. Thus, the passive city will choose the course of action to close for an outcome valuation of five, knowing the raiders will choose the course of action to avoid. The aggressive city will choose the course of action to ambush, for an outcome valuation of six, knowing the raiders will choose the course of action to siege rather than choose the course of action to avoid. For the raiders, the expected outcome valuation in Sub-game 2 for the course of action to avoid is 3.2 and is greater than the expected outcome valuation of 2.8 for choosing the course of action to lay siege.

This yields the Bayesian equilibrium^{1,2,3} in which the passive city will choose the course of action to close and the raiders will choose the course of action to avoid and where the aggressive city will choose the course of action to ambush and the raiders will choose the course of action to avoid. Twenty percent of the time the raiders will face a passive city and have an outcome valued at four and 80 percent of the time they will face an aggressive city and have an outcome valued at three, as seen in Figure 7.4.

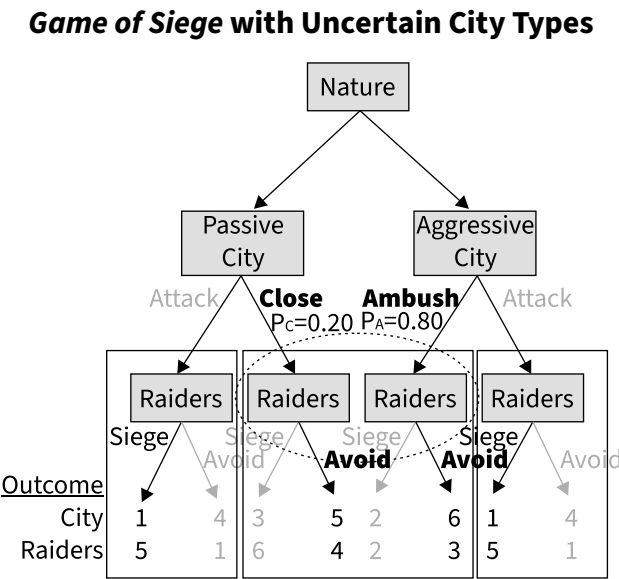


Figure 7.4: Resolving the three sub-games up, using the raiders’ expected outcome values in the Sub-game 2, the raiders will choose to avoid if they are not attacked and will choose to lay siege if they are attacked. The passive city will choose to close and the aggressive city will choose to ambush.

In the second way of formulating the *Game of Siege with Uncertainty*, the raiders face a city that may choose the course of action to ambush or to close. In this game, the city and the raiders face outcome valuations presented in Table 6.3 and reproduced in Table 7.2. These outcome valuations yielded the extensive form of the *Game of Siege with Uncertainty* presented in Figure 7.5.

Table 7.2: *Game of Siege with Uncertainty Outcomes*

The course-of-action choice pairs for the *Game of Siege with Uncertainty*, coupled with their outcomes and their preference valuations as presented in Table 6.3 from Chapter VI.

CITY'S COA	RAIDERS' COA	OUTCOME	CITY'S VALUE	RAIDERS' VALUE
Close	Siege	No Casualties, Ransom Paid	3	6
Close	Avoid	No Casualties, No Ransom	5	4
Ambush	Siege	Many Casualties Some Looting	2	2
Ambush	Avoid	Few Casualties, No Ransom	6	3
Attack	Siege	Many Casualties, City Looted	1	5
Attack	Avoid	Some Casualties, No Ransom	4	1

Second Game of Siege with Uncertainty

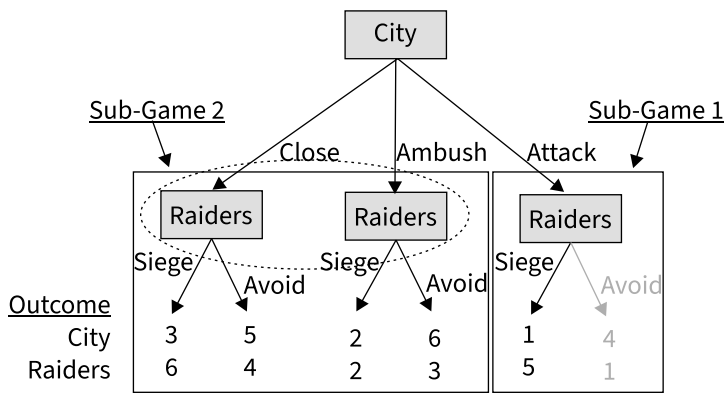


Figure 7.5: The extensive form of the second *Game of Siege with Uncertainty* with Sub-game 1 resolved upward.

In this game, Sub-game 1 resolves up, indicated in Figure 7.5, as it did in Figure 6.15. That is, if the raiders find themselves in Sub-game 1, knowing the city’s forces are arranged to attack, the raiders will always choose the course of action to lay siege for an outcome valuation of five, which is greater than the outcome valuation of one if they choose the course of action to avoid.

If the raiders find themselves in Sub-game 2, where the city’s course-of-action choices to close and to ambush are observationally equivalent to the raiders’, the raiders’ information set has two decision nodes. The raiders cannot discern at which decision node they are. In the previous chapter, we transformed Sub-game 2 to its equivalent simultaneous-play game and solved that game to its Nash equilibria. There were two equilibria: the city chooses the course of action to close and the raiders choose the course of action to siege with outcome valuations of three to the city and six to the raiders, or the city chooses the course of action to ambush and the raiders choose the course of action to avoid with an outcome valuation of six to the city and an outcome valuation of three to the raiders.

To resolve Sub-game 2 up one level, the raiders need more information about which of the two decision nodes they are at within that information set. In this iteration of the *Game of Siege with Uncertainty*, we assume the raiders understand that the city is a rational actor and their probability distribution across their course-of-action choice in Sub-game 2 is in the city's best interests. That is, the raiders know that the city always randomly selects between the course of action to close and the course of action to ambush with a specific probability distribution that gives the city the best outcome possible. The raiders will use their knowledge of the city's incentives to form a belief over the city's probability distribution to inform their course-of-action choice in Sub-game 2.

Looking at the city's outcome valuations in Sub-game 2, the course-of-action path where the city chooses to close and the raiders choose to avoid (close, avoid) yields an outcome valuation of five to the city, and the course-of-action path (ambush, avoid) yields an outcome valuation of six to the city. The course-of-action paths (close, siege) and (ambush, siege) yield outcome valuations of three and two, respectively, to the city. It is unambiguously better for the city if the raiders choose to avoid conflict regardless of the city's course-of-action choice. Knowing the city will act in its best interest, it is reasonable for the raiders to believe the city will choose a probability distribution across their course-of-action choices to close and to ambush, to make the course-of-action choice to ambush the most attractive one for the raiders.

It is reasonable for the raiders to believe that the city will choose the probability with which it chooses its course of action to close (p_{close}) and will choose the probability with which it chooses its course of action to ambush ($p_{\text{ambush}} = 1 - p_{\text{close}}$), so the raiders' expected outcome valuation of choosing their course of action to avoid is greater than the raiders' expected outcome valuation of choosing to lay siege. Thus, the raiders believe that the city's probability distribution across its courses of action to close and to ambush meet the following criteria:

$$\begin{aligned}
& EV_{\text{raiders}}(\text{Avoid}) \geq EV_{\text{raiders}}(\text{Siege}), \\
& (\rho_{\text{close}} \times 4) + (\rho_{\text{ambush}} \times 3) \geq (\rho_{\text{close}} \times 6) + (\rho_{\text{ambush}} \times 2), \\
& (\rho_{\text{close}} \times 4) + ((1 - \rho_{\text{close}}) \times 3) \geq (\rho_{\text{close}} \times 6) + ((1 - \rho_{\text{close}}) \times 2), \\
& (3 + \rho_{\text{close}}) \geq (2 + 4 \times \rho_{\text{close}}), \text{ or } 3 \times \rho_{\text{close}} \leq 1, \text{ or} \\
& \rho_{\text{close}} \leq 1/3.
\end{aligned}$$

Thus, the raiders have a reasonable belief that the city will choose their course of action to close at a frequency less than one-third of the time.

Just for the purpose of an example in this iteration of the *Game of Siege with Uncertainty*, let us say the raiders believe that the city will choose the course of action to close its gates 20 percent of the time (the city chooses to close with a probability $\rho_{\text{close}} = 0.20 \leq 1/3$), thus the raiders believe the city chooses the course of action to ambush 80 percent of the time (the city chooses to ambush with a probability $\rho_{\text{ambush}} = 0.80$). Given the raiders' beliefs over the city's probability distribution across the two decision nodes the raiders face in their information set in Sub-game 2, the game the raiders believe they are in is indicated in Figure 7.6. This scenario gives a parallel structure to the two-city type game introduced earlier in this chapter.

The raiders, believing that the city chooses to close with a probability of $\rho_{\text{close}} = 0.20$, and chooses to ambush with a probability of $\rho_{\text{ambush}} = 0.80$, then the raiders can expect an outcome valuation of six 20 percent of the time and an outcome valuation of two 80 percent of the time. This yields an expected outcome value to the raiders of

$$EV_{\text{city}}(\text{Avoid}) = (0.20 \times 6) + (0.80 \times 2) = 2.8.$$

Using exactly the same thought process, if the raiders choose the course of action to avoid in Sub-game 2, believing that there is a $\rho_{\text{close}} = 0.20$ probability the city has chosen to close and that there is a $\rho_{\text{ambush}} = 0.80$ probability the city has chosen to ambush, then the raiders have the following expected outcome valuation:

$$EV_{\text{city}}(\text{Avoid}) = (0.20 \times 4) + (0.80 \times 3) = 3.2.$$

Second Game of Siege with Uncertainty

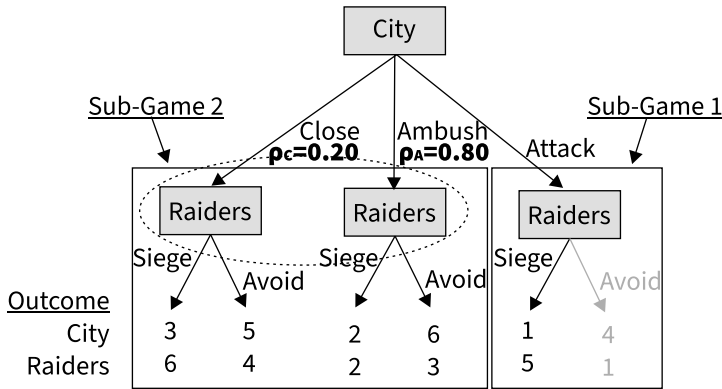


Figure 7.6: In Sub-game 2 the raiders believe the city will close with $p_{\text{Close}} = 0.20$ and set an ambush with $p_{\text{Ambush}} = 0.80$. The raiders use their knowledge of the city's probability distribution to resolve Sub-game 2 up to the next level by calculating the expected value of the outcomes associated with siege or avoid.

The raiders, when they find themselves in Sub-game 2, with their information set, only perceive two courses of action (siege and avoid) that lead to their respective expected outcome valuations. When the raiders evaluate the expected outcome valuations associated with their course-of-action choice, the raiders will choose the course of action to avoid with an expected outcome valuation of 3.2 over choosing the course of action to lay siege with an expected outcome valuation of 2.8, as indicated in Figure 7.7.

This allows Sub-game 2 to resolve up one level where the city knows that the raiders will choose to avoid given the city's probability distribution across their course-of-action choices of to close and to ambush.

Second Game of Siege with Uncertainty

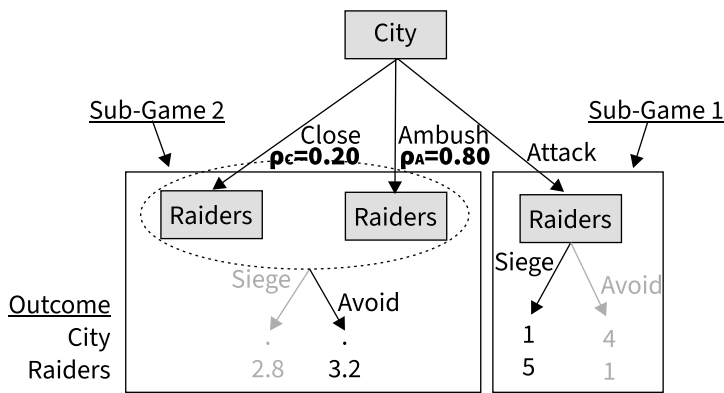


Figure 7.7: Given the raiders’ expected outcomes, the raiders will choose the course of action to avoid in Sub-game 2, resolving the game up to the next level.

The city calculates the expected value of its outcomes in Sub-game 2, as indicated in Figure 7.8. Given their probability distribution across closing and ambushing, conditioned on being in Sub-game 2, the city expects an outcome valuation of 2.2 if the raiders choose to lay siege, and an outcome valuation of 5.8 if the raiders choose to avoid. Because the city’s expected outcome valuation of 5.8 in Sub-game 2 when the raiders choose their course of action to avoid exceeds the outcome valuation of one in Sub-game 1, in which the raiders choose their course of action to siege, this *Game of Siege with Uncertainty* resolves to an equilibrium, with the city’s expected outcome value of 5.8 in Sub-game 2, when the raiders choose to avoid exceeding the outcome value of one in Sub-game 1, in which the raiders choose the course of action to siege. The city will never choose the course of action to attack, as indicated in Figure 7.8.

This thought process leads us to the Bayesian Equilibrium, where, if cities always choose to mix their courses of action to close and to ambush with the probability distribution ($p_{\text{Close}} = 0.20$, $p_{\text{Ambush}} = 0.80$), the raiders, knowing their enemy’s probability distribution, will choose to avoid. The city earns an expected outcome value of 5.8, and the raiders earn an expected outcome value of 3.2.

Second Game of Siege with Uncertainty

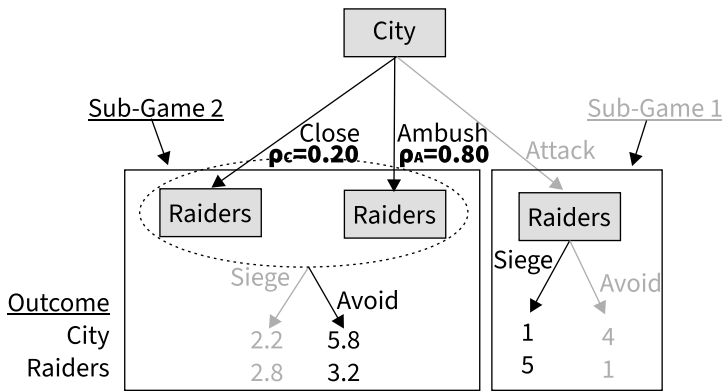


Figure 7.8: Here is the Bayesian Equilibrium, where the city always chooses to mix its courses of action to close to and ambush and the raiders choose to avoid.

Before concluding that this outcome to the game is reasonable, it is fair to question whether all probability distributions, where $p_{\text{Close}} \leq \frac{1}{3}$, are likely to be chosen by the city. To address this issue, given that the raiders understand the city will choose $p_{\text{Close}} \leq \frac{1}{3}$, making the raiders' expected outcome valuation highest if they choose their course of action to avoid, is there a value for p_{Close} that maximizes the city's expected outcome? In short, is there a "best" p_{Close} for the city? This is a question the raiders need to address when forming their belief of the probability distribution that the city will choose in Sub-game 2.

The city's expected outcome valuation if it chooses $p_{\text{Close}} = \frac{1}{3}$ is $EV_{\text{city}} = (\frac{1}{3} \times 5) + (\frac{2}{3} \times 6) = 5.67$. The city's expected outcome valuation if it chooses $p_{\text{Close}} = 0.20$ is $EV_{\text{city}} = (0.20 \times 5) + (0.80 \times 6) = 5.8$, as calculated in the earlier scenario. And the city's expected outcome valuation if it chooses $p_{\text{Close}} = 0$ is $EV_{\text{city}} = (0 \times 5) + (1 \times 6) = 6$. Given this sequence of expected outcome valuations available to the city, it is reasonable for the raiders to believe the city will choose the probability distribution $p_{\text{close}} = 0$ and $p_{\text{ambush}} = 1$ when the raiders find themselves in Sub-game 2, as indicated in Figure 7.9.

Second Game of Siege with Uncertainty

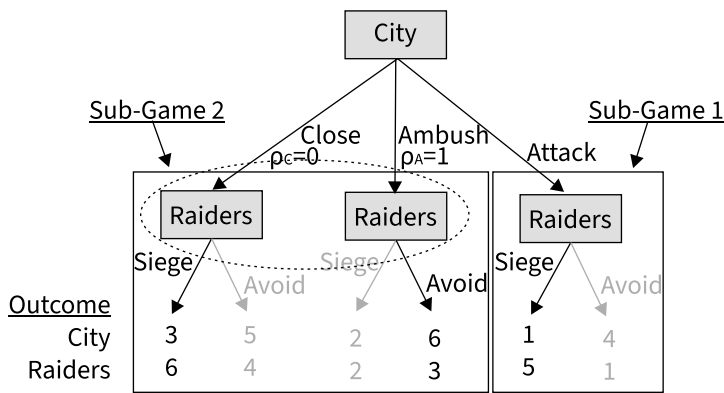


Figure 7.9: In Sub-game 2 the raiders believe the city will close with $p_{\text{Close}} = 0$ and set an ambush with $p_{\text{Ambush}} = 1$. The raiders use their knowledge of the city's probability distribution to resolve Sub-game 2 up to the next level by calculating the expected value of the outcomes associated with siege or avoid.

In this scenario, the raiders believe that the city will choose the course of action to close with probability $p_{\text{Close}} = 0$. Given this belief, the raiders, when considering the game, are essentially pruning away the city's course of action to close, because, the raiders think, a rational city would never choose the course of action to close. If the city sees that the raiders would choose their course of action to siege as a best response to the city's choosing the course of action to attack in Sub-game 1, and that the raiders would choose their course of action to avoid as their best response in Sub-game 2, then the city will choose its course of action to ambush. This yields the equilibrium detailed in Figure 7.10, where the city chooses the course of action to ambush, and the raiders choose to avoid if in Sub-game 2 and choose to siege if the city chooses to attack.

Second Game of Siege with Uncertainty

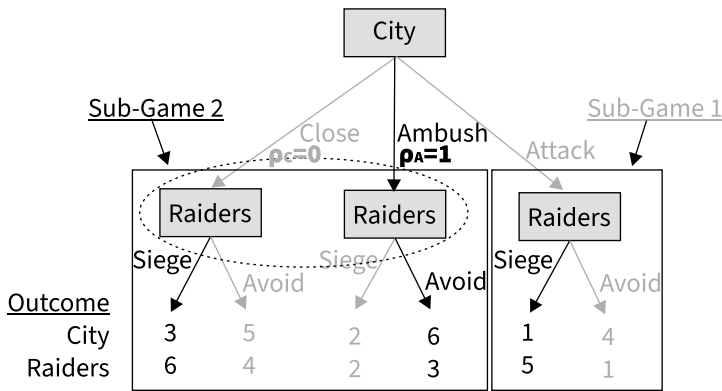


Figure 7.10: In this scenario of the game, where the raiders believe the city will always choose the courses of action to ambush in Sub-game 2, the raiders choose their course of action to avoid when in Sub-game 2. This scenario resolves to an equilibrium where the city will choose its course of action to ambush and the raiders will choose their course of action to avoid, as highlighted.

To employ the process of looking forward and reasoning back in a decision environment where at least one of the players has multiple decision nodes in an information set, the players must know the distribution of conditional probability weightings across the decision nodes inside of the information set. With this probability information, players can evaluate the expected value of each course-of-action choice and choose the course of action that will give the largest expected value, allowing players to resolve the game up a level. This process continues until, through this modified form of backwards induction, the path of reasonable course-of-action choices is outlined to solve the path to the Bayesian Perfect Equilibrium.⁴

The next chapter will present a more involved second case study of the game *Operation Overlord/Fortress Europe with Uncertainty*. The extended game theory structure and the new solution techniques introduced in this chapter will be applied to the historical battle, although simplified. These two relatively simple sequential-play

decision environments develop a relatively straightforward thought process to resolve sequential-play games with uncertainty to their respective equilibrium. The insights developed in Chapter VI and Chapter VII are applied to the Allied invasion of Normandy during World War II. This battle was partially developed in Chapter II and Chapter III. In the next chapter, the game *Operation Overlord/ Fortress Europe* is presented with intentional misdirection on the part of the Allies.

CHAPTER VIII

CASE 2: OPERATION OVERLORD/ FORTRESS EUROPE

The location where we will engage the enemy must not become known to them. If it is not known, then the positions which they must prepare to defend will be numerous. If the positions the enemy prepares to defend are numerous, then the forces we will engage will be few.

SUN TZU, *ART OF WAR*

In the game *Operation Overlord/Fortress Europe*, as introduced in Chapters II and III, and as modified in Figure 8.1, Germany's defense of Western Europe involves fixed coastal positions as well as mobile forces, including Panzers that could either be held inland as reserves in a second line of defense or be committed to the coasts as part of the first line of defense. In the previous game as described, the Allied forces do know the German defense posture and whether the mobile tank divisions are held inland or deployed at the coast. Depending upon the defense posture chosen by Germany, the Allied forces then choose to attack at Calais or at Normandy. If the German mobile forces were held inland, the probability of the Allies gaining a foothold on continental Europe or being repelled by German defenses would show the Allied forces with a better chance of succeeding on either beach than if the mobile defenses were positioned forward at the coast. This is because there would be some delay before the German mobile forces could move to and engage with the Allies' invading force. The longer the Allies could delay Germany from committing its mobile forces against their invasion force, the greater the probability of the Allies gaining a foothold on continental Europe.

In the game *Operation Overlord/Fortress Europe*, as indicated in Figure 8.1, Germany could make the course-of-action choice to position mobile forces inland under three circumstances. These circumstances are in coordination with building fixed defenses at Calais, building fixed defenses at Normandy, or building fixed defenses evenly across Calais and Normandy. Prior to invasion, the Allies know whether Germany has committed its mobile forces to the coast as part of the front line of defense or whether Germany has held its mobile forces in reserve inland. In a new game, *Operation Overlord/Fortress Europe with Uncertainty*, at each of the Allies' three decision nodes

Operation Overlord/Fortress Europe

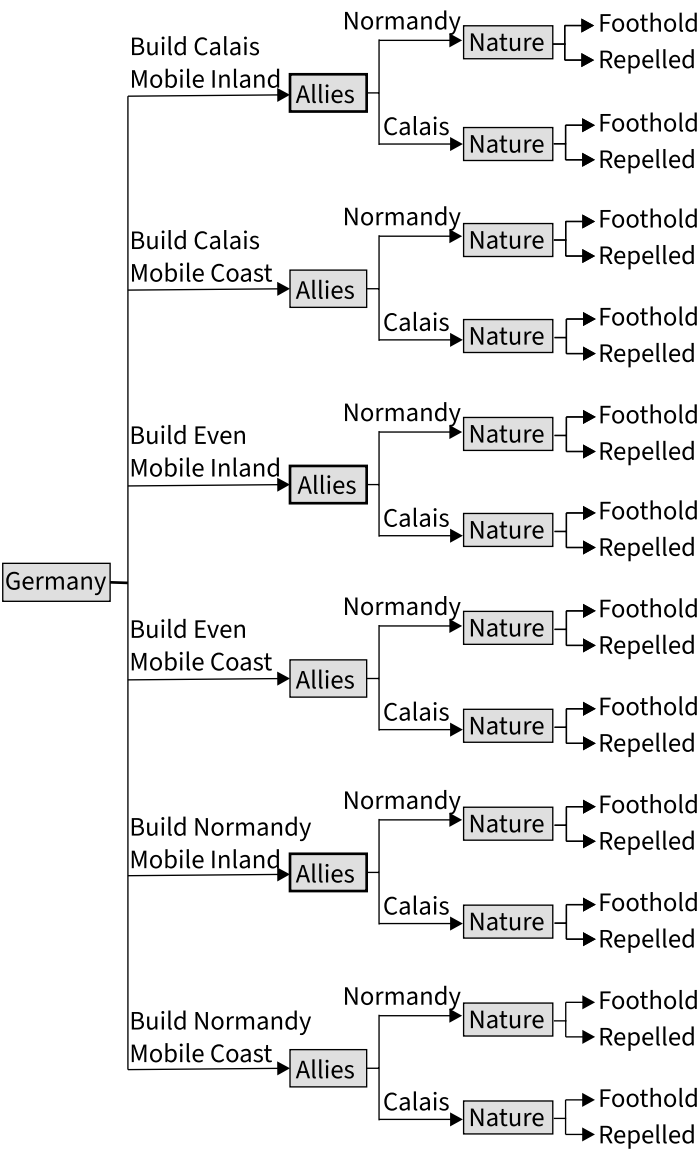


Figure 8.1: The *Operation Overlord/Fortress Europe* game from Chapter III modified to reflect only two possible battle outcomes: the Allies gain a foothold in Europe or the Germans repel the attack.

(highlighted in Figure 8.1), in which Germany's course-of-action choice has its mobile defenses inland, the Allies can now engage in a deception that may delay Germany from committing its mobile inland forces to the beach the Allied forces are attacking.

Thus, at the three Allied decision nodes, in which Germany has held its mobile forces inland, the Allies will now have four course-of-action choices. As indicated in Figure 8.1, the Allies could engage in a direct and overt invasion at Calais or in a direct and overt invasion at Normandy. In addition to the two course-of-action choices from the original game, the Allies can now choose the course of action to pretend to attack Normandy but really attack at Calais, as well as choose the course of action to pretend to attack Calais but really attack at Normandy. Their two new course-of-action choices will be called as follows: feign Normandy attack Calais and feign Calais attack Normandy. For the course-of-action choice to feign Normandy and attack Calais, the Allies will present a force posture that implies an attack on Normandy, even though their intended target is Calais. Thus, to Germany, the Allies' course-of-action choice to attack Calais and the course-of-action choice to feign at Calais and attack at Normandy are initially indistinguishable. Similarly, to Germany, the Allies' course-of-action choice to attack the beaches at Normandy and the course-of-action choice to feign attacking Normandy and instead attack at Calais are also initially indistinguishable.

Germany's inability to distinguish between a direct attack and a feigned attack is represented by the two information sets illustrated in Figure 8.2. Information Set 1 indicates that Germany cannot tell whether the Allies have chosen the course of action to attack Normandy or the course of action to feign attacking Normandy and instead attack Calais until *after* Germany has chosen its course of action either to move its mobile forces to Normandy or hold its mobile forces inland. If the Allies have chosen a course of action from Information Set 1, for example, either to attack the beaches at Normandy or to feign at Normandy and actually attack Calais, the Germans then choose the course of action to maintain their mobile forces inland and

wait for the situation to resolve, or they choose the course of action to move their mobile forces to Normandy. Germany has similar course-of-action choices if the Allies choose a course of action in Information Set 2.

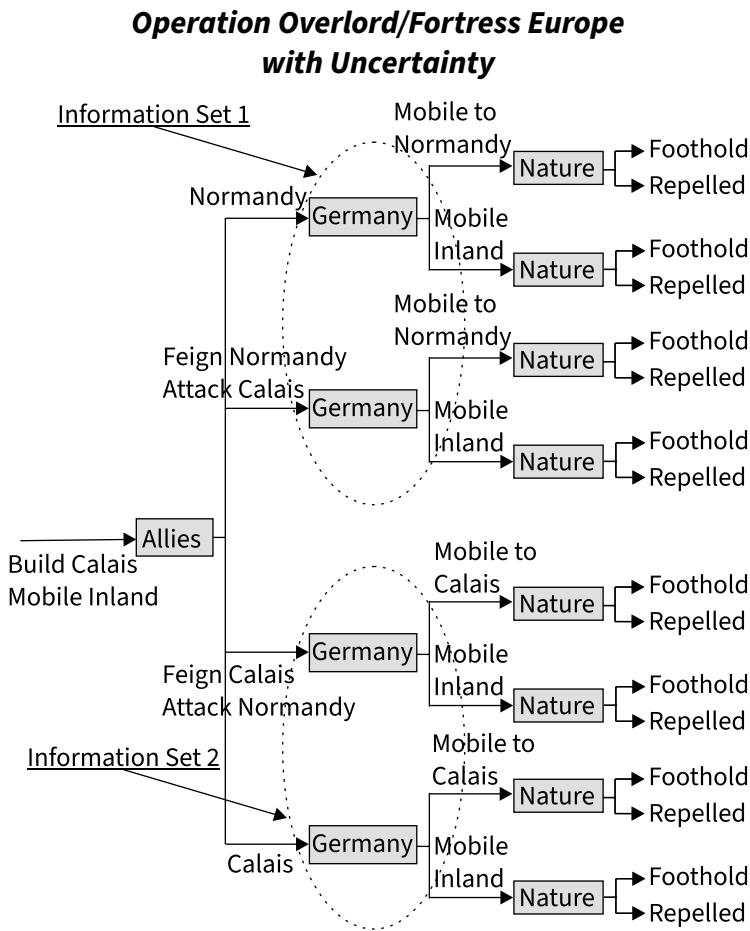


Figure 8.2: This is the branch of the game where the Germans have chosen to build coastal defenses asymmetrically in Calais and hold mobile forces inland. The Allies can choose overt attacks at Calais or Normandy or engage in force posture deception. The information sets indicate Germany’s inability to differentiate a feint from a direct attack. This table connects the Allied and German course-of-action pairs to their respective outcome probabilities.

Once the Allies and Germany have moved through the game tree as indicated in Figure 8.2, conflict ensues and Nature resolves *Operation Overlord/Fortress Europe with Uncertainty* to one of the two possible outcomes: either the Allies gain a foothold, or Germany repels the invasion. The probabilities by which Nature resolves the outcome of battle are determined by the doctrinal rule of thumb that a force attacking a prepared defensive position needs a three-to-one force advantage for a 50 percent probability of victory, as discussed in earlier chapters. When the attacking-to-defending ratios change, the probability of the attacking force's victory also changes. In this scenario, the probability of the Allies gaining a foothold on continental Europe changes. To be consistent with the game *Operation Overlord/Fortress Europe* developed in Chapter II and presented in Figure 2.8, the force ratios presented there are used in Table 8.1, and the probabilities are adjusted to match the circumstance.

As seen in Table 8.1, because Germany has asymmetrically built fixed defenses favoring Calais, an Allied attack on Normandy with German mobile forces held inland gives the Allies approximately a three-to-one force advantage against German forces at Normandy. The Allies, therefore, have a 50 percent probability of gaining a foothold on continental Europe at Normandy. This probability distribution across the outcomes is expressed in columns three and four in row two of Table 8.1.

Nature yields one of two outcomes: either the Allies gain a foothold on continental Europe or the Allied invasion is repelled. The Allies prefer to gain a foothold to the outcome of being repelled and thus give the outcome for a foothold a higher outcome valuation than the outcome valuation for being repelled:

$$V_{\text{Allies}}(\text{foothold}) = 1 > V_{\text{Allies}}(\text{repelled}) = 0.$$

Similarly, the Germans prefer the outcome in which their enemies are repelled and give it a higher outcome valuation than the outcome valuation for the Allies gaining a foothold:

$$V_{\text{Germany}}(\text{repelled}) = 1 > V_{\text{Germany}}(\text{foothold}) = 0.$$

Thus, the outcome foothold yields the outcome valuation pair of $(V_{\text{Germany}}(\text{foothold}), V_{\text{Allies}}(\text{foothold})) = (0, 1)$, and the outcome repelled yields the outcome valuation pair of $(V_{\text{Germany}}(\text{repelled}),$

$V_{Allies}(\text{repelled})) = (1,0)$. These outcome valuations, with Germany’s outcome valuation listed first and the Allies’ outcome valuation listed second, are seen in Table 8.1, just below the probability of the potential outcome.

Table 8.1: Courses of Action to Outcomes and Probabilities

Given that the Germans have chosen to build coastal defenses asymmetrically in favor of Calais and have chosen to hold their mobile forces in reserve inland, and the Allies choose to engage in force posture deception, this table connects the Allied and German course-of-action pairs to their respective outcome probabilities and outcome valuations.

Branch: Germany Builds Asymmetrically at Calais, Mobile Forces Inland			
Players’ Courses of Action		Outcome Probabilities and Valuations	
ALLIES	GERMANY	FOOTHOLD	REPELLED
Attack Normandy	Mobile Normandy	0.20 (0, 1)	0.80 (1, 0)
Attack Normandy	Mobile Inland	0.50 (0, 1)	0.50 (1, 0)
Feign Normandy Attack Calais	Mobile Normandy	0.60 (0, 1)	0.40 (1, 0)
Feign Normandy Attack Calais	Mobile Inland	0.40 (0, 1)	0.60 (1, 0)
Feigh Calais Attack Normandy	Mobile Calais	0.65 (0, 1)	0.35 (1, 0)
Feigh Calais Attack Normandy	Mobile Inland	0.55 (0, 1)	0.45 (1, 0)
Attack Calais	Mobile Calais	0.10 (0, 1)	0.90 (1, 0)
Attack Calais	Mobile Inland	0.25 (0, 1)	0.75 (1, 0)

With the information in Table 8.1, the branch of *Operation Overlord/Fortress Europe* in which Germany has chosen the course of action to build coastal fixed forces asymmetrically in Calais and keep its mobile forces in reserve inland (as displayed in Figure 8.2) can be updated, as indicated in Figure 8.3. With the players' outcome

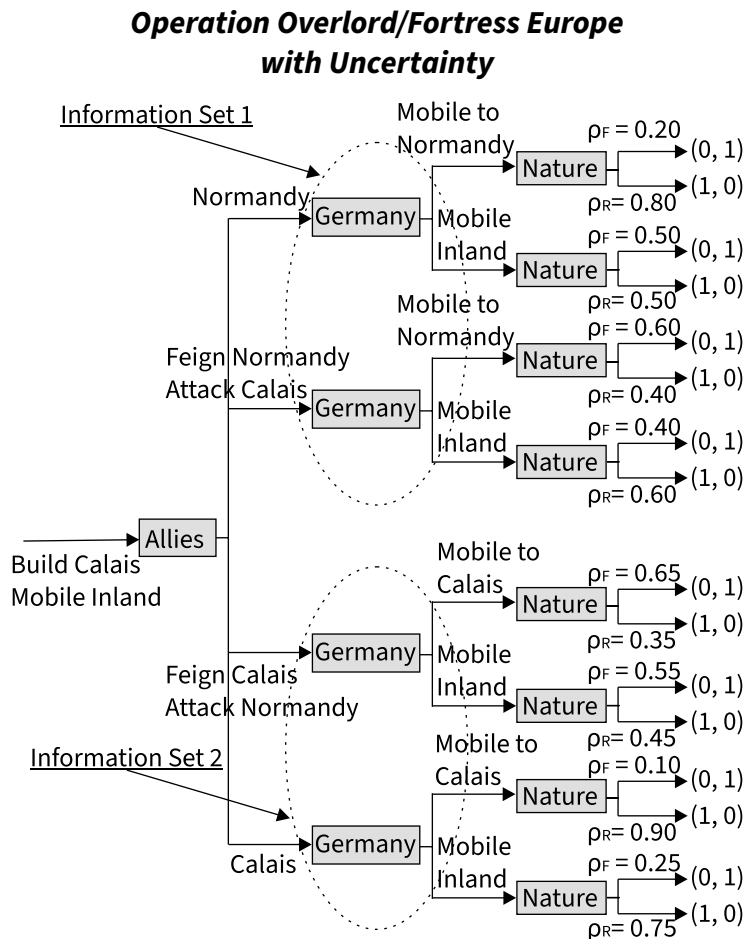


Figure 8.3: In this branch of the game, the Germans have chosen to build coastal defenses asymmetrically in Calais and hold mobile forces inland. The information sets indicate Germany's inability to differentiate a feint from a direct attack. This table connects the Allied and German course-of-action pairs to their respective outcome probabilities (p_{Foothold} and p_{Repelled}) and respective outcome valuations (V_{Germany} , V_{Allies}).

valuations (V_{Germany} , V_{Allies}), as well as the associated probabilities (ρ_{Foothold} or ρ_{Repelled}), Nature will resolve to those respective outcomes and give us the information necessary to resolve this branch of the game. Starting at the outcome valuations and reasoning back up through the nonstrategic player, Nature, the expected outcomes of Germany's course-of-action choices are either to move its mobile forces to the actual or perceived Allied attack point or to hold its mobile forces inland for further clarification. These expected outcome valuations take the form of

$$\begin{aligned} EV_{\text{Germany}} &= \rho_F \times V_{\text{Germany}}(\text{Foothold}) + \rho_R \times V_{\text{Germany}}(\text{Repelled}), \text{ or} \\ EV_{\text{Germany}} &= 0.20 \times 0 + 0.80 \times 1 = 0.80, \end{aligned}$$

for Germany on the top branch of Figure 6.7. Similarly, it takes the form

$$\begin{aligned} \text{of } EV_{\text{Allies}} &= \rho_F \times V_{\text{Allies}}(\text{Foothold}) + \rho_R \times V_{\text{Allies}}(\text{Repelled}), \text{ or} \\ EV_{\text{Allies}} &= 0.20 \times 1 + 0.80 \times 0 = 0.20 \end{aligned}$$

for the Allies on the same branch in Figure 6.7. This figure represents the game when we add the probabilities and outcome valuations for each of Nature's branches.

The expected outcome valuations for each of the choice sets in the branch of the game displayed in Figure 8.3 are inserted, representing how Nature resolves the game to outcomes, on average. These expected outcome valuations inform players of what to expect if the game proceeds down each of the potential pathways. Figure 8.4 represents the branch of *Operation Overlord/Fortress Europe with Uncertainty* in which Germany has chosen to asymmetrically build fixed coastal defenses favoring Calais and has chosen to hold its mobile forces inland, when we resolve it up one level to Germany's decision nodes.

Along this branch of the game as displayed in Figure 8.4, Germany cannot discern between a direct attack or a feigned attack by the Allies. Thus, each information set in the branch of the game also defines the sub-game Germany faces when the Allies choose to attack or feign an attack. Germany's Sub-game 1 is coincidental with Germany's Information Set 1, while Sub-game 2 is coincidental with Information

Set 2. Germany's two sub-games are also represented in Figure 8.4. Although the Germans can know whether they are in Sub-game 1 or Sub-game 2, they cannot know whether the Allies have chosen the course of action to attack Normandy or to feign at Normandy and instead attack Calais, as in Sub-game 1 and, similarly, Sub-game 2.

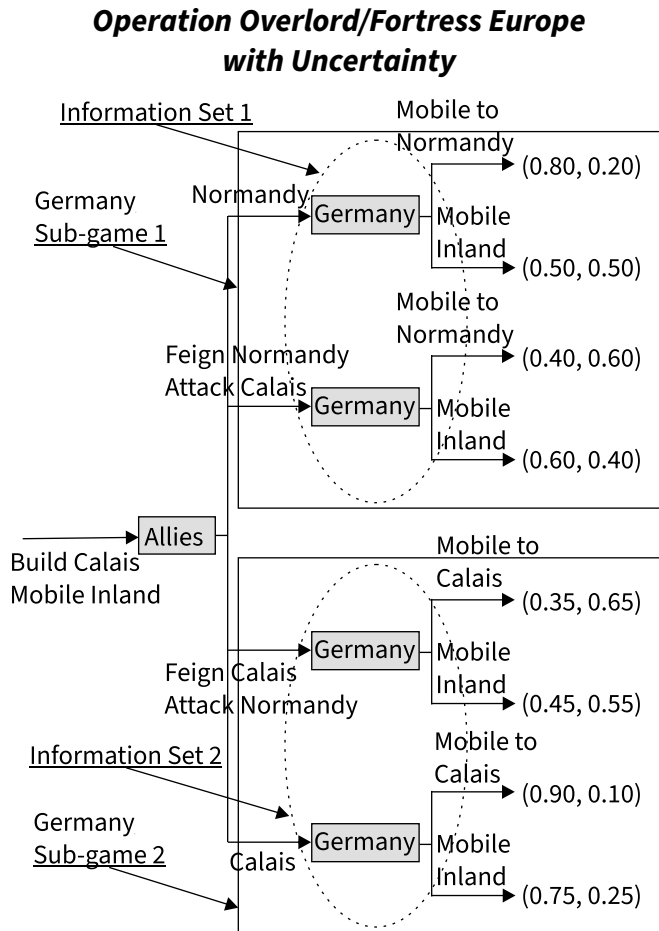


Figure 8.4: In this branch of the game, the Allied and German course-of-action pairs are linked to their respective expected outcome valuations (EV_{Germany} , EV_{Allies}). At this point in the game's solution, Germany has two sub-games coincidental with their two information sets.

As we have done with the branch of the game in which Germany has chosen the course of action to build at Calais with its mobile forces inland, we must analyze the other two branches of the game to develop them. For the case in which Germany chooses the course of action to build fixed forces evenly across invasion sites with mobile forces inland over the course of action build, in order to build fixed forces at Normandy with mobile inland, the expected outcome valuation analysis is highlighted in Figure 8.2. The course-of-action pairs chosen by the Allies and Germany in each of these remaining two game branches must be mapped to their corresponding outcome valuations and outcome probabilities. This analysis is furnished in Table 8.1. Similar information is presented in Table 8.2 and Table 8.3 for the other two branches of the game.

Taking the information presented in Table 8.2 and applying the expected outcome valuation process demonstrated in Figure 8.4, we develop the branch of the game where Germany has chosen to build fixed coastal defenses evenly between Calais and Normandy and has chosen to hold its mobile forces in reserve inland. The expected outcome valuations for this branch of the game are derived in Figure 8.5 by using the outcome probabilities as determined by the nonstrategic player Nature and the outcome valuations, for both Germany and the Allies, of the Allied forces gaining a foothold on continental Europe and of the Allied forces being repelled.

Similarly, Figure 8.6 shows the expected outcome valuations for Germany and the Allies for the branch of the game where Germany has chosen to build fixed coastal defenses asymmetrically toward Normandy and has chosen to hold its mobile forces in reserve inland.

When we take Figures 8.4, 8.5, and 8.6 and insert them into their respective branches of the game represented in Figure 8.2, the full extensive form of *Operation Overlord/Fortress Europe with Uncertainty* is generated and presented in Figure 8.7. Figure 8.7 also includes the outcome probabilities—the probability that the Allies gain a foothold on continental Europe (ρ_F), the probability that Germany repels

the Allied invasion force (p_R), and the outcome valuations for Germany and the Allies for each of Nature's possible battle resolutions (V_{Germany} , V_{Allies}).

Table 8.2: Courses of Action to Outcomes and Probabilities

This table connects the Allied and German course-of-action pairs to their respective outcome probabilities and outcome valuations, given that the Germans have chosen to build coastal defenses symmetrically across Calais and Normandy and have chosen to hold their mobile forces in reserve inland, and the Allies have chosen to engage in force posture deception.

Branch: Germany Builds Evenly at Calais/Normandy, Mobile Forces Inland			
Players' Courses of Action		Outcome Probabilities and Valuations	
ALLIES	GERMANY	FOOTHOLD	REPELLED
Attack Normandy	Mobile Normandy	0.25 (0, 1)	0.75 (1, 0)
Attack Normandy	Mobile Inland	0.50 (0, 1)	0.50 (1, 0)
Feign Normandy Attack Calais	Mobile Normandy	0.60 (0, 1)	0.40 (1, 0)
Feign Normandy Attack Calais	Mobile Inland	0.55 (0, 1)	0.45 (1, 0)
Feigh Calais Attack Normandy	Mobile Calais	0.60 (0, 1)	0.40 (1, 0)
Feigh Calais Attack Normandy	Mobile Inland	0.55 (0, 1)	0.45 (1, 0)
Attack Calais	Mobile Calais	0.25 (0, 1)	0.75 (1, 0)
Attack Calais	Mobile Inland	0.50 (0, 1)	0.50 (1, 0)

As is done in the three branches of the game in which Germany chooses to keep its mobile forces in reserve inland, the three branches of the game delineated in Figure 8.7—where Germany has chosen to

Table 8.3: Courses of Action to Outcomes and Probabilities

This table connects the Allied and German course-of-action pairs to their respective outcome probabilities and outcome valuations, given that the Germans have chosen to build coastal defenses asymmetrically in favor of Normandy and have chosen to hold their mobile forces in reserve inland, and the Allies have chosen to engage in force posture deception.

Branch: Germany Builds Asymmetrically at Calais/Normandy, Mobile Forces Inland			
Players' Courses of Action		Outcome Probabilities and Valuations	
ALLIES	GERMANY	FOOTHOLD	REPELLED
Attack Normandy	Mobile Normandy	0.10 (0, 1)	0.90 (1, 0)
Attack Normandy	Mobile Inland	0.25 (0, 1)	0.75 (1, 0)
Feign Normandy Attack Calais	Mobile Normandy	0.55 (0, 1)	0.45 (1, 0)
Feign Normandy Attack Calais	Mobile Inland	0.55 (0, 1)	0.45 (1, 0)
Feigh Calais Attack Normandy	Mobile Calais	0.60 (0, 1)	0.40 (1, 0)
Feigh Calais Attack Normandy	Mobile Inland	0.40 (0, 1)	0.55 (1, 0)
Attack Calais	Mobile Calais	0.20 (0, 1)	0.80 (1, 0)
Attack Calais	Mobile Inland	0.50 (0, 1)	0.50 (1, 0)

posture its mobile forces forward with the fixed coastal defenses—need to be resolved up past the nonstrategic player Nature. To do so, the expected outcome valuations for Germany and for the Allies need to be determined for each of the German/Allied choice pairs in each

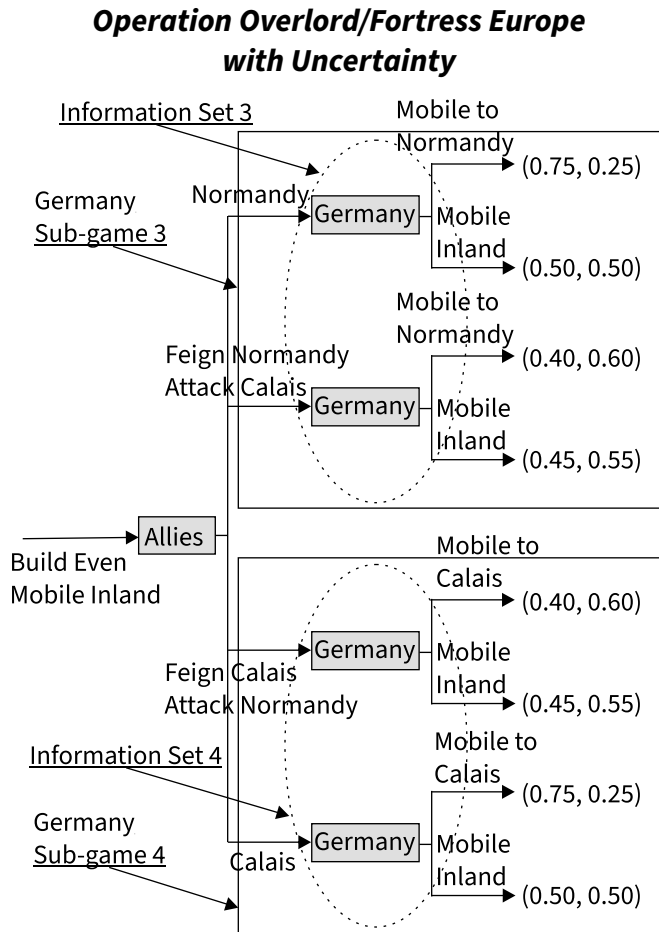


Figure 8.5: In this branch of the game, the Allied and German course-of-action pairs are linked to their respective expected outcome valuations (EV_{Germany} , EV_{Allies}). At this point in the game's solution, Germany has two sub-games coincidental with their two information sets.

of the three remaining branches. These expected outcome valuations take the form of

$$\begin{aligned} EV_{\text{Germany}} &= \rho_F \times V_{\text{Germany}}(\text{Foothold}) + \rho_R \times V_{\text{Germany}}(\text{Repelled}), \text{ or} \\ EV_{\text{Germany}} &= 0.50 \times 0 + 0.50 \times 1 = 0.50 \end{aligned}$$

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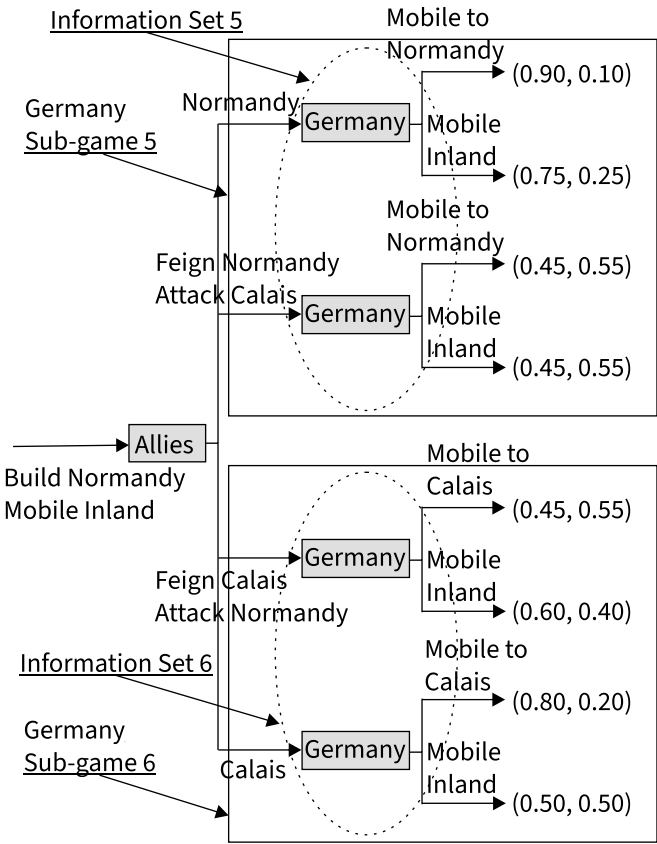


Figure 8.6: In this branch of the game, the Allied and German course-of-action pairs are linked to their respective expected outcome valuations (EV_{Germany} , EV_{Allies}). At this point in the game's solution, Germany has two sub-games coincidental with their two information sets.

for Germany on the top branch of Figure 8.7. Similarly, it takes the form of

$$\begin{aligned} EV_{\text{Allies}} &= \rho_F \times V_{\text{Allies}}(\text{Foothold}) + \rho_R \times V_{\text{Allies}}(\text{Repelled}), \text{ or} \\ EV_{\text{Allies}} &= 0.50 \times 1 + 0.50 \times 0 = 0.50 \end{aligned}$$

for the Allies on the same branch in Figure 8.7. The expected outcome valuations are reported (EV_{Germany} , EV_{Allies}) in Figure 8.9.

Also, in Figure 8.8, the Allies are faced with three distinct subgames coincident with the three branches of the game where Germany has positioned its mobile forces at the coast. When choosing a course of action, Germany can look forward and determine what response a rational opponent would make. Assuming the Allies would choose a course of action that would give them a higher expected outcome valuation, in the top Allied sub-game, where Germany has chosen to build asymmetrically in favor of Calais and has positioned its mobile forces at the coast, the Allies can choose the course of action to attack Normandy with an expected outcome valuation of $EV_{\text{Allies}}(\text{Normandy}) = 0.50$, or they can choose to attack Calais with an expected outcome valuation of $EV_{\text{Allies}}(\text{Calais}) = 0.10$. Since

$$EV_{\text{Allies}}(\text{Normandy}) = 0.50 > 0.10 = EV_{\text{Allies}}(\text{Calais}),$$

Germany can assume the Allies will choose to attack Normandy. Following that logic, Germany can assume that the Allies are indifferent to choosing between the courses of action to attack Normandy or to attack Calais in the Allies' second sub-game, in which Germany has chosen the course of action to build its fixed coastal defenses evenly with mobile forces along the coast. In the Allies' third sub-game, in which Germany has chosen the course of action to build at Normandy with mobile forces along the coast, the Allies' highest expected outcome valuation is attained if they choose the course of action to attack Calais.

Having analyzed the easiest subgames up to the top level of the game, we are left with the three branches of the game presented in Figures 8.4, 8.5, and 8.6. In these branches, Germany faces subgames that have information sets containing two decision nodes. It

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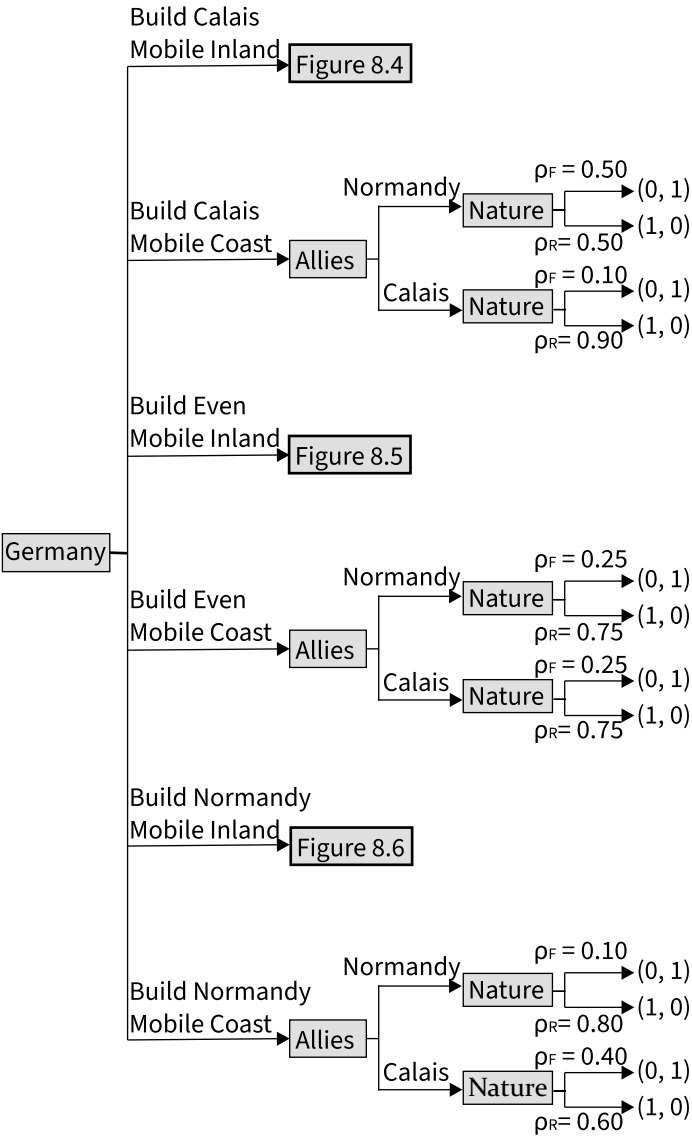


Figure 8.7: The full representation of the *Operation Overlord/Fortress Europe with Uncertainty* game, with Allied force posture deception on the decision nodes where Germany chose to keep its mobile forces in reserve inland.

is situations like this that Bayesian Perfect Nash Equilibrium analysis is designed to address. But to resolve sub-games as indicated in Figure 8.8, Germany must have an additional piece of information. The Germans must know the probability at a particular decision node within the information set. That is, they must know the probability that the Allies are engaging in a direct attack at, for example, Normandy (ρ_{AN}), and the probability that the Allies are feigning an attack at Normandy only to attack at Calais (ρ_{FN}), as delineated in Information Set 1 in Figure 8.9. As historian Steven Ambrose has explained, to have a chance to gain a defensible foothold on continental northern Europe, the Allies needed to “immobilize the Panzer divisions through trickery—fooling the Germans not only in advance of the attack but making them believe the real thing was a feint.”¹

The German high command up to Hitler himself was very certain the Allies would attack at Pas-de-Calais. To sweeten the target, Germany built the launch sites for the V-1 cruise missiles and V-2 ballistic missiles nearby.² Given this level of confidence on the part of Germany that the “real” attack would target Calais, in Sub-game 1 of Figure 8.9, if Germany sees an attack at Normandy, it will assign a probability that the attack is real at $\rho_{AN} = 0.30$ and the probability that the attack at Normandy is a feint, with the real attack coming at Calais at $\rho_{FN} = 0.70$. For symmetry in reasoning in Sub-game 2 of Figure 8.9, we will say that Germany will assign the likelihood of seeing an attack at Calais with a probability of $\rho_{AC} = 0.70$, and the probability that the attack at Calais is a feint and the real attack is coming at Normandy with $\rho_{FC} = 0.30$.

In Sub-game 1 of Figure 8.9, the Germans do not know whether they are at the decision node in which the Allies have chosen the course of action to attack Normandy or the decision node in which the Allies have chosen to feign Normandy and attack Calais. The Germans have two course-of-action choices: They can choose the course of action to place their mobile forces inland until they have further clarification of the situation, or they can choose to move their mobile forces to the coast to meet the invaders at Normandy.

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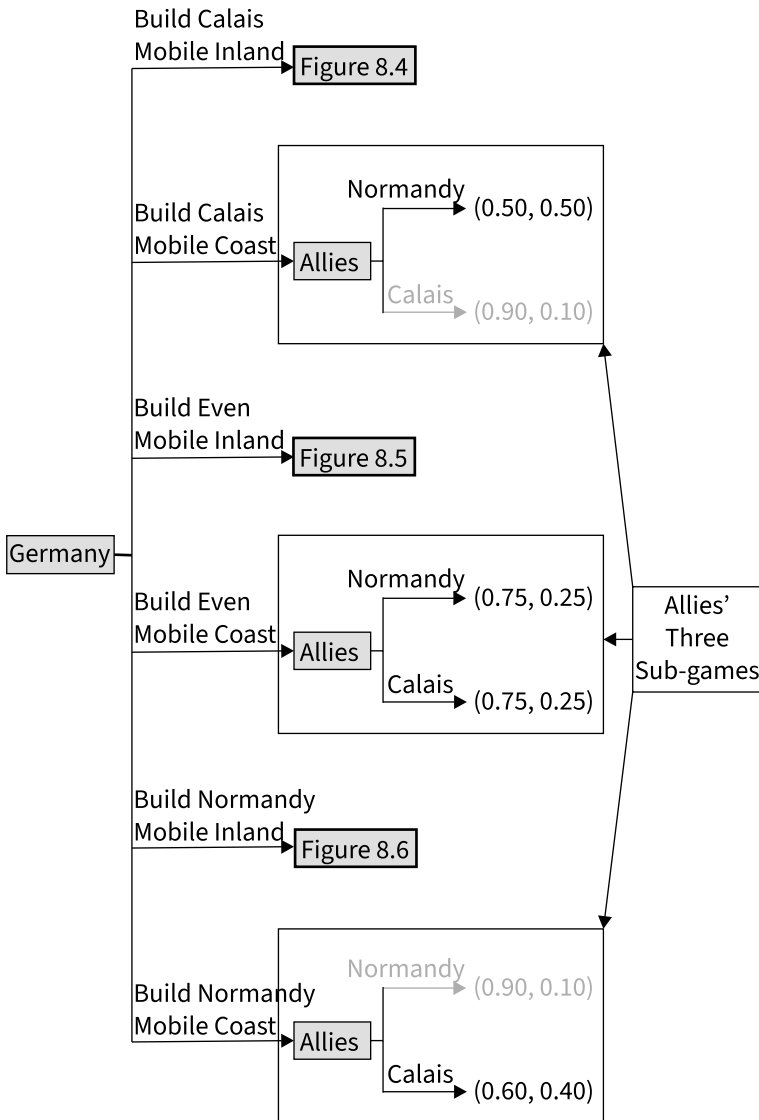


Figure 8.8: The *Operation Overlord/Fortress Europe with Uncertainty* game resolving the expected outcome valuations (EV_{Germany} , EV_{Allies}) for the branches of the game where Germany chooses to position mobile forces forward with fixed coastal defenses.

Operation Overlord/Fortress Europe with Uncertainty

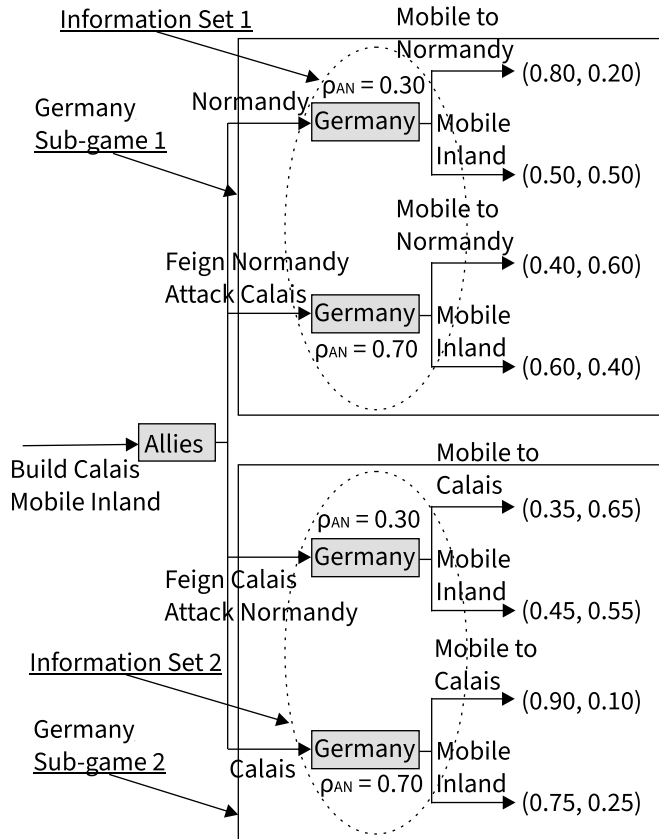


Figure 8.9: The Germans must determine the probability that they are at a particular decision node within each of their information sets so they can resolve the two sub-games up to the next level of game play.

If the Allies attack Normandy outright, then Germany is at the top decision node in Sub-game 1 and will achieve an expected outcome valuation $EV_{\text{Germany}}(\text{Mobile Normandy}) = 0.80$ if Germany chooses the course of action to move mobile forces to Normandy. If the Allies feign an attack at Normandy to attack Calais, then Germany is at the

bottom decision node in Sub-game 1 and will achieve an expected outcome valuation $EV_{\text{Germany}}(\text{Mobile Normandy}) = 0.20$ if Germany chooses the course of action to move mobile forces to Normandy. If Germany assumes the probability of an attack on Normandy is $\rho_{\text{AN}} = 0.30$ and the probability of a feint on Normandy, followed by an attack at Calais, is $\rho_{\text{FN}} = 0.70$, then it will expect an outcome valuation of choosing to move mobile forces to Normandy of

$$EV_G(\text{Mobile to Normandy}) = \rho_{\text{AN}} \times V_G(\text{Mobile to Normandy}) + \rho_{\text{FM}} \times V_G(\text{Mobile to Normandy}), \text{ or}$$

$$V_G(\text{Mobile to Normandy}) = 0.70 \times 0.80 + 0.30 \times 0.40 = 0.68.$$

Using the exact same thought process, Germany's expected outcome valuation for choosing the course of action to place mobile forces inland when in Sub-game 1 is

$$EV_G(\text{Mobile Inland}) = \rho_{\text{AN}} \times V_G(\text{Mobile Inland}) + \rho_{\text{FM}} \times V_G(\text{Mobile Inland}), \text{ or}$$

$$EV_G(\text{Mobile Inland}) = 0.70 \times 0.50 + 0.30 \times 0.60 = 0.53.$$

Expected outcome valuations can be determined in Sub-game 2 the same way. The Germans' expected outcome valuations for the branch of the game where they choose the course of action to build asymmetrically in Calais and hold their mobile forces inland are reported in Figure 8.10.

Given the expected outcome valuations for each course-of-action choice in Sub-games 1 and 2 of the branch of the game portrayed in Figure 8.10, Germany can make informed course-of-action choices. If Germany finds itself in Sub-game 1, the course-of-action choice to bring mobile forces to Normandy has an expected outcome valuation of

$$EV_G(\text{Mobile to Normandy}) = 0.68 > 0.53 = EV_G(\text{Mobile Inland}),$$

which is the expected outcome valuation of choosing the course of action mobile inland. So given the higher expected outcome valuation of choosing the course of action to bring mobile forces to Normandy, Germany will always make that course-of-action choice in Sub-game 1.

Operation Overlord/Fortress Europe with Uncertainty

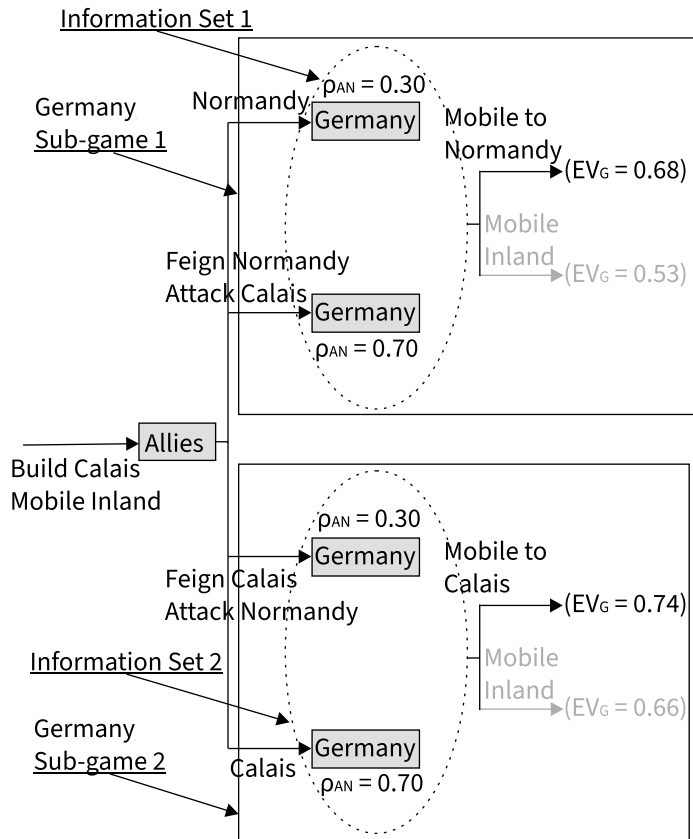


Figure 8.10: The Germans' expected outcome valuations for their course-of-action choices in each of the sub-games for this branch of the game. In addition, their best course of action if they find themselves in Sub-game 1 is to move mobile forces to Normandy. Likewise, in Sub-game 2, their best choice is to move mobile forces to Calais.

By the same logic, Germany has the expected outcome valuations in Sub-game 2, in which

$$EV_G(\text{Mobile to Calais}) = 0.75 > 0.66 = EV_G(\text{Mobile Inland}),$$

so it will always choose the course of action to bring mobile forces to Calais. Thus, in the branch of *Operation Overlord/Fortress Europe with Uncertainty* in which Germany has chosen the course of action to build Calais mobile inland, given its assumption about the probability that the Allies are feigning an attack or are actually attacking Calais or Normandy, a rational Germany will choose the course of action to mobilize to Normandy if in Sub-game 1 and choose the course of action to mobilize to Calais if in Sub-game 2, as highlighted in Figure 8.10.

Once Germany's response to being in Sub-game 1 and Sub-game 2 in this branch of the game is determined, the solving of the game resolves up one more level. Now the Allies choose their course of action given Germany's course-of-action choices. From Figure 8.10, Information Set 1, we see that the Allies know Germany will choose the course of action to move to Normandy, and in Information Set 2, they know that Germany will choose the course of action to move its mobile forces to Calais. The sub-game the Allies are playing is represented in Figure 8.11. Reasoning through this branch of the game, if the Allies choose the course of action to attack Normandy, as seen in Information Set 2, Germany will choose to move its mobile forces to Normandy, and the Allies will receive the expected outcome valuation of 0.20, as indicated in Figure 8.11. The Allies' best expected outcome valuation, given the German response in this branch of the game, is achieved when the Allies choose the course of action to feign to Calais and attack Normandy. This course-of-action choice puts Germany in Information Set 2, where the best expected outcome valuation is achieved when that nation chooses to mobilize to Calais. This path in the game leads to Germany receiving an expected outcome valuation of $EV_{\text{Germany}} = 0.35$ and the Allies receiving an expected outcome valuation of $EV_{\text{Allies}} = 0.65$. This course-of-action choice path is highlighted in Figure 8.11.

The branches of the game in which Germany chooses the course of action to build its fixed coastal forces evenly while positioning its mobile forces inland and the course of action to build its fixed coastal

forces at Normandy while positioning its mobile forces inland can also be resolved by applying the thought process that led to Figure 8.11. In Figure 8.6, we see it is possible for Germany to believe that the Allies are predisposed to feign to Normandy and attack Calais. As Information Set 2 in Figure 8.11 illustrates, the Germans may believe that the

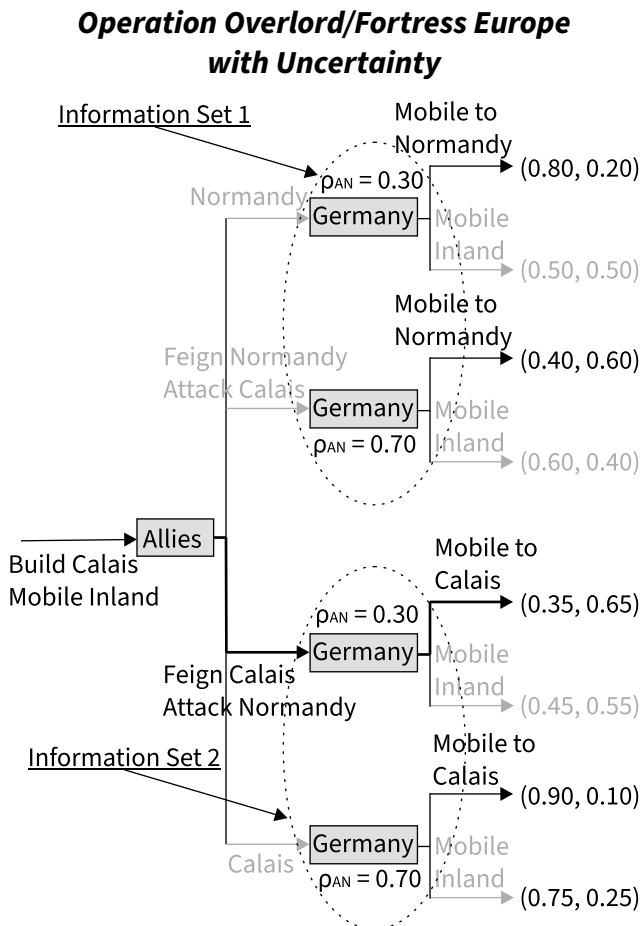


Figure 8.11: The Allies, knowing the German course-of-action choice in each of their information sets, now choose the course of action that is most beneficial to them.

probability of that happening is $\rho_{FN} = 0.70$, while they may also believe that the probability that the Allies will attack Normandy is $\rho_{AN} = 0.30$. They may alternatively believe there is a probability of $\rho_{AC} = 0.70$ that the Allies will attack Calais, or feign to Calais and attack Normandy with a probability of $\rho_{FC} = 0.30$. If Germany accepts these probabilities, then in Sub-Game 3, as shown in Figure 8.12, Germany will choose the course of action to mobilize forces to Normandy, with an expected outcome valuation of $EV_G = 0.65 > 0.49 = EV_G$, which is the expected outcome valuation for the course-of-action choice to mobilize inland. Likewise, Germany will choose the course of action to move mobile forces to Calais if they find themselves in Sub-game 4.

When we resolve the branch of the game in which the Germans have chosen the course of action to build their fixed coastal forces evenly while holding their mobile forces inland, as shown in 8.12, and where they will then choose the course of action to deploy mobile forces to Normandy when in Sub-game 3 and mobilize to Calais in Sub-game 4, we can consider the Allies' course-of-action choice as displayed in Figure 8.13. Here, the Allies are indifferent to choosing between the course-of-action choice to feign to Normandy and attack Calais and the course-of-action choice to feign to Calais and attack Normandy. With either course-of-action choice, the Allies will achieve their highest expected outcome valuation of $EV_{Allies} = 0.60$, given Germany's response to being in Information Set 3 or Information Set 4, as indicated in Figure 8.13.

The last branch of the game that needs to be solved is the branch in which Germany chooses the course of action to build Normandy mobile inland, as portrayed in Figure 8.13. We can calculate Germany's expected outcome valuations in Sub-games 5 and 6 using Germany's perceived probabilities of the Allies' force posture as $\rho_{AN} = 0.30$ and $\rho_{FN} = 0.70$ in Sub-game 5 and $\rho_{FC} = 0.30$ and $\rho_{CA} = 0.70$ in Sub-game 6. A rational German course-of-action choice in Sub-game 5 would then be mobile to Normandy with an expected outcome valuation $EV_G = 0.75 > 0.66 = EV_G$, which is the expected outcome valuation of choosing the course of action mobile inland. Similarly, Germany's

course-of-action choice in Sub-game 6 is mobile to Calais with an expected outcome valuation $EV_G = 0.73 > 0.63 = EV_G$, which is the expected outcome valuation of choosing the course of action mobile inland.

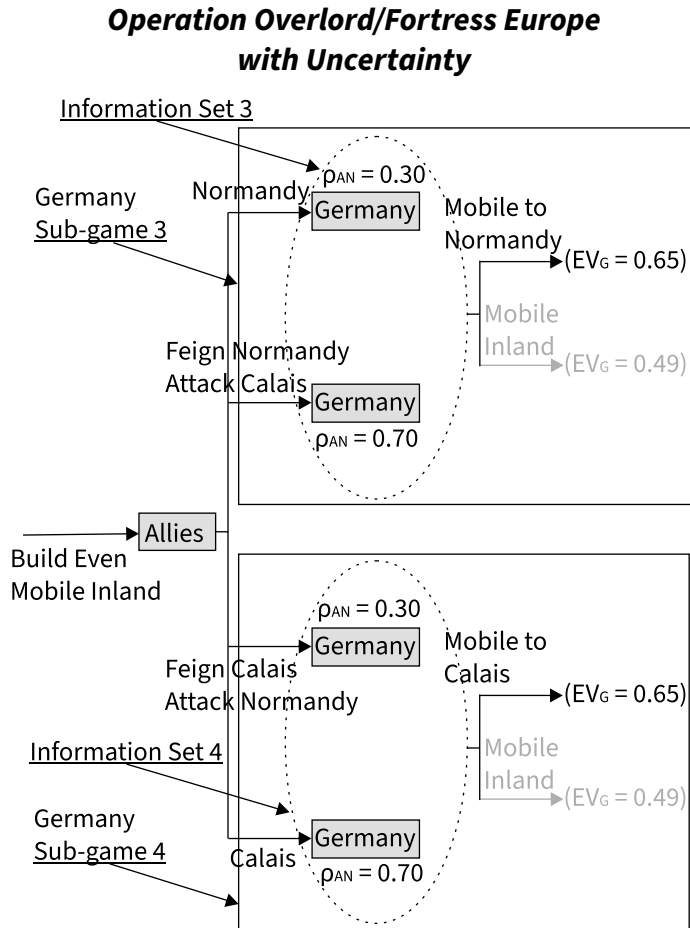


Figure 8.12: The Germans' expected outcome valuations for their course-of-action choices in each of the sub-games and Germany's best course of action for each of the sub-games.

Having determined Germany’s course-of-action choices as shown in Figure 8.14, when Germany faces Information Set 5 and Information Set 6, we can turn our analysis up the game tree and determine the Allies’ best course-of-action choices given Germany’s, as shown in Figure 8.15. The Allies have four course-of-action choices in this branch of the game: to attack at Normandy; to feign at Normandy; to attack at Calais; and to feign at Calais.

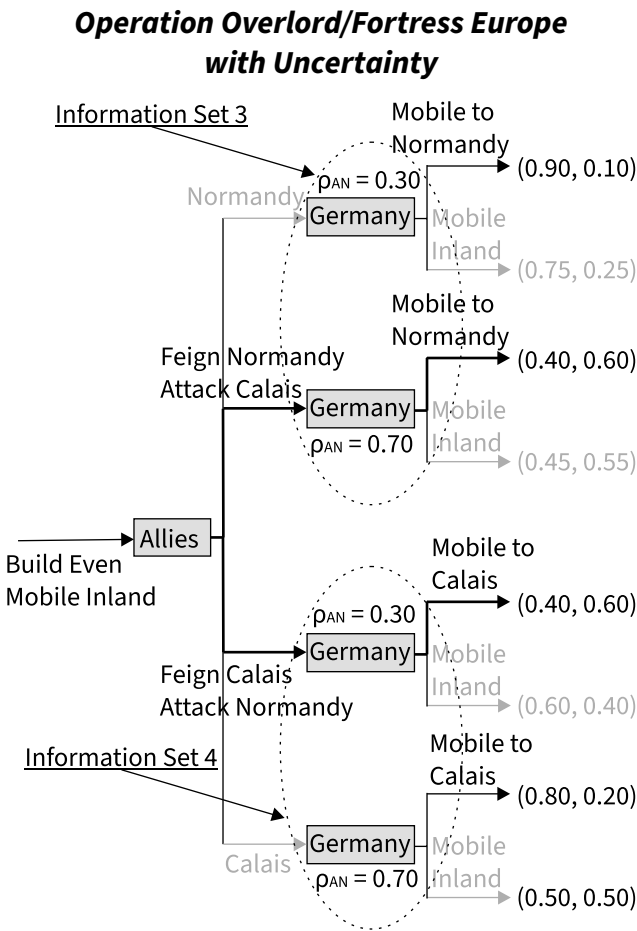


Figure 8.13: The Allies, knowing the Germans’ course-of-action choice in each of their information sets, now choose the course of action that is most beneficial to them.

attack Calais, creating Germany's Information Set 5; to feign at Calais and attack at Normandy; and attack at Calais, creating Germany's Information Set 6. When in Information Set 5, the Allies know that Germany will choose the course of action to deploy its mobile forces to Normandy. Thus, the Allies will achieve an expected outcome valuation when choosing the course of action to attack at Normandy of

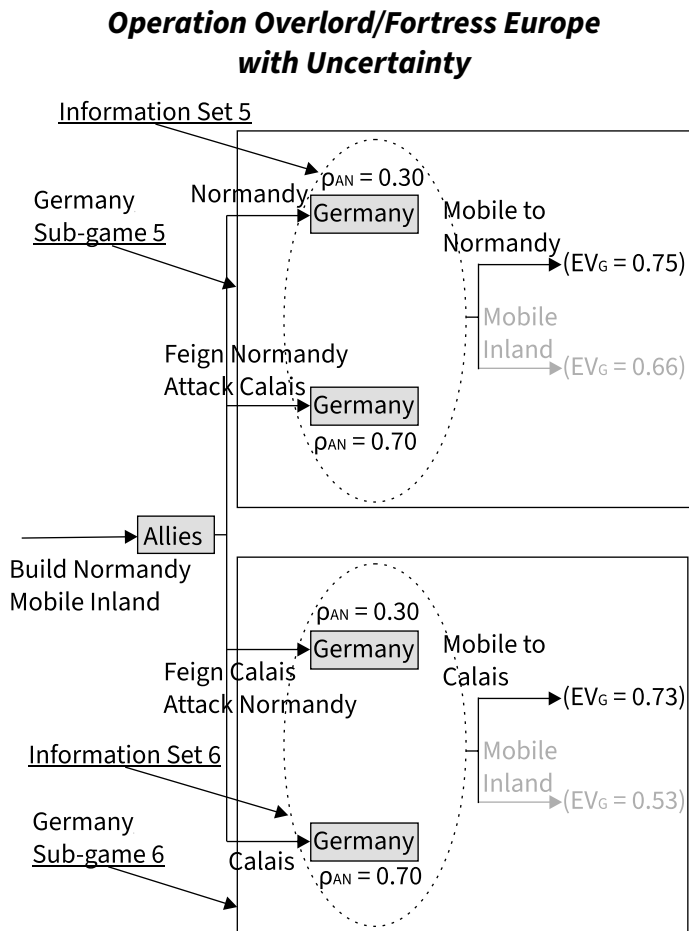


Figure 8.14: The Germans' expected outcome valuations for Sub-game 5 and Sub-game 6, as well as their best course-of-action choice in each sub-game.

$EV_A = 0.10 < 0.60 = EV_A$, the expected valuation the Allies achieve when choosing the course of action feign Normandy attack Calais. When in Information Set 6, the Allies know that Germany will choose the course of action to deploy its mobile forces to Calais. Thus, the Allies will achieve an expected outcome valuation when choosing the course of action to feign to Calais and attack at Normandy of

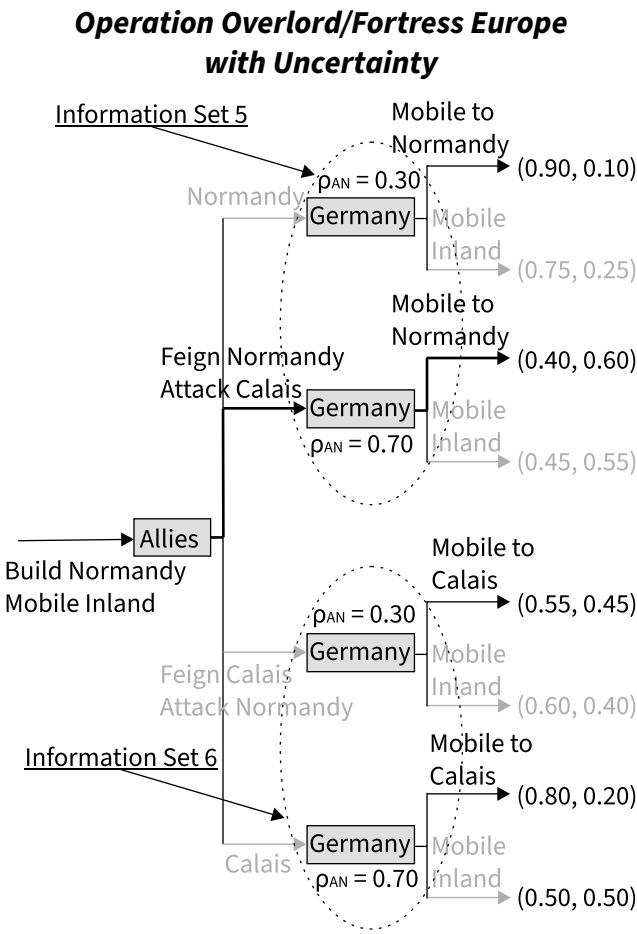


Figure 8.15: The Allies, knowing the German course-of-action choice in each of their information sets, now choose the course of action that is most beneficial to them.

$EV_A = 0.45 > 0.20 = EV_A$, which is the expected valuation the Allies achieve when choosing course of action attack Calais. Given this set of expected outcome valuations, the Allies will choose the course of action to feign at Normandy and attack at Calais for their best expected outcome valuation of $EV_A = 0.60$, as illustrated in Figure 8.15.

In Figure 8.8, *Operation Overlord/Fortress Europe with Uncertainty* is presented resolving three of the branches of the game up through the Allies' sub-games. The other three branches with information sets containing a pair of German decision nodes have been resolved up to the Allies' sub-game as well. These outcomes—indicated in Figures 8.11, 8.13, and 8.15—can now be inserted into the game tree represented in Figure 8.8. This gives a full representation of the course-of-action sequences that Germany and the Allies will find advantageous to follow.

As an aside, keep in mind that *Operation Overlord/Fortress Europe with Uncertainty* played out as illustrated in the highlighted gray off-equilibrium path in Figure 8.16. Here we see that Germany chose the course of action to build at Calais with its mobile forces inland; then the Allies chose the course of action to feign at Calais and attack at Normandy; and finally, Germany chose the course of action to place its mobile forces inland. This choice path leads to Germany's expected outcome valuation $EV_G = 0.45$ and the Allies' expected outcome valuation $EV_A = 0.55$, if the actual game is well represented by the model presented here.

We have solved the sub-games of each branch of the full game as illustrated in Figure 8.16. Each sub-game up to the Allies' first decision node leaves Germany choosing among six initial course-of-action choices. Thus, from the top decision node of the game, Germany is choosing among six course-of-action choices: to build fixed coastal defenses at Calais with mobile forces inland; to build fixed coastal defenses at Calais with mobile forces on the coast; to build fixed coastal defenses evenly with mobile forces inland; to build fixed coastal defenses evenly with mobile forces on the coast; to build fixed coastal defenses at Normandy with mobile forces inland; and to build

fixed coastal defenses at Normandy with mobile forces on the coast. Each of the Germans' course-of-action choices lead down the game tree to the expected outcome valuations in which both Germany and the Allies are choosing to follow courses of action to maximize their respective expected outcome valuations.

If Germany chooses its first course of action to build fixed coastal defenses at Calais with mobile forces inland, the game plays out to the outcome with the expected outcome valuations of $(EV_G, EV_A) = (0.35, 0.65)$. Similarly, if Germany chooses its third course of action to build fixed coastal defenses at Calais with mobile forces at Calais, the game plays out with the expected outcome valuations of $(EV_G, EV_A) = (0.50, 0.50)$. If Germany chooses its second course of action, to build fixed coastal defenses evenly with mobile forces inland, the game plays out to two possible end states, both with the expected outcome valuations of $(EV_G, EV_A) = (0.40, 0.60)$. If Germany chooses its fourth course of action, to build fixed coastal defenses evenly with mobile forces on the coast, the game plays out to two possible end states, both with the expected outcome valuations of $(EV_G, EV_A) = (0.75, 0.25)$. If Germany chooses its fifth course of action, to build fixed coastal defenses at Normandy with mobile forces inland, the game plays out with the expected outcome valuations of $(EV_G, EV_A) = (0.60, 0.60)$. And finally, if Germany chooses its sixth course of action, to build fixed coastal defenses at Normandy with mobile forces at Normandy, the game plays to the outcome with the expected outcome valuations of $(EV_G, EV_A) = (0.60, 0.40)$.

Given Germany's expected outcome valuation for each of its six course-of-action choices, its expected outcome valuation is highest if it chooses to build evenly with mobile forces on the coast. If Germany chooses that course of action, it will receive an expected outcome valuation of $EV_G = 0.75$. The Bayesian equilibrium is the course-of-action choice path starting with Germany's choice to build evenly with mobile forces on the coast, followed by the Allies' choice either to attack at Normandy or attack at Calais. The two Bayesian equilibria are highlighted in Figure 8.16.

Operation Overlord/Fortress Europe with Uncertainty

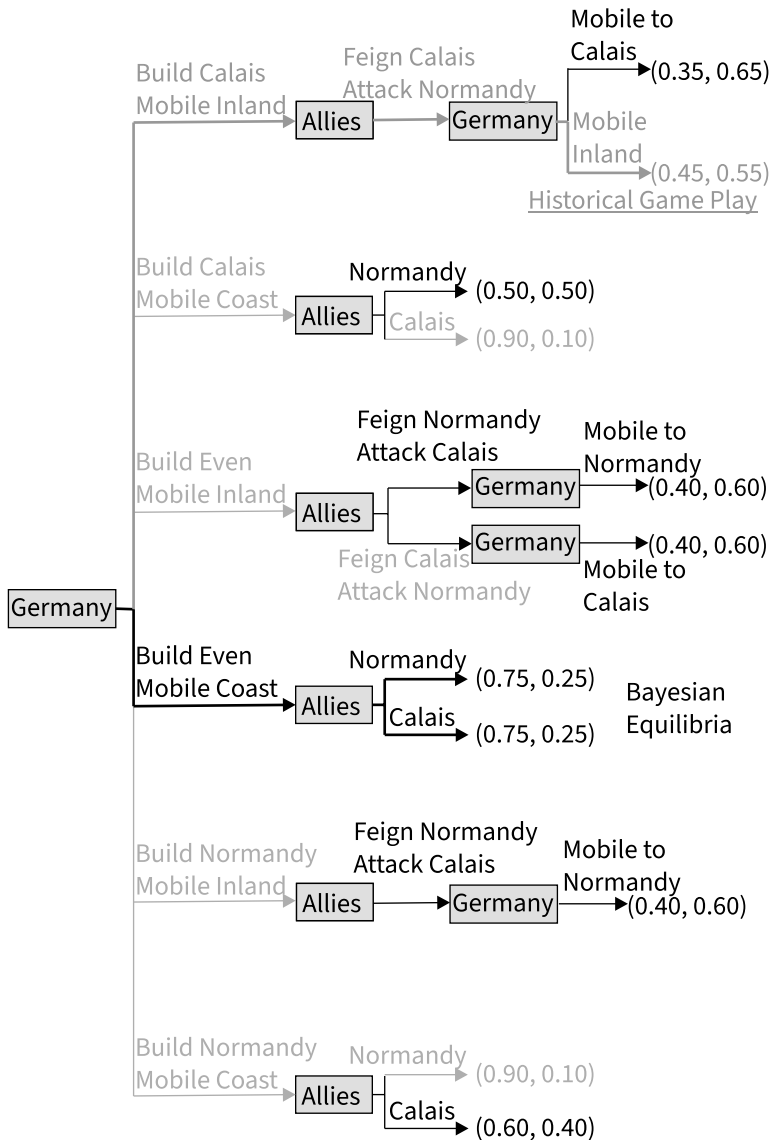


Figure 8.16: The Operation Overlord/Fortress Europe with Uncertainty game resolving the expected outcome valuations (EV_{Germany} , EV_{Allies}) for all the branches of the game. The Bayesian Equilibrium is the dark highlighted choice path.

As indicated in the first iteration of *Operation Overlord/Fortress Europe*, Field Marshal Rommel has recommended that the game's sub-game perfect equilibrium be Germany's European strategy, in particular that the mobile Panzer divisions need to be deployed forward. Rommel's advice has been overridden by Field Marshal von Rundstedt, who believes the Panzer divisions need to be held in reserve as a counter-offense. In this iteration of the game, in which the Allies engage in deceptive force postures, the Bayesian equilibrium again reenforces the tactical wisdom of Field Marshal Rommel.

To find the Bayesian equilibrium in this chapter, the analysis relies on Germany having prior knowledge of the probability distribution the Allies use to determine the propensity with which they choose the course of action to feign to Normandy and attack at Calais and the course of action to attack at Normandy. It is not clear whether or not Germany has a good understanding of that probability distribution. The question arises as to whether there is something about the structure of the game that would allow Germany to derive the Allies' rational probability distribution. In the next chapter, the prior probability distribution knowledge requirement is relaxed and a mechanism as to how those probabilities are emergent from the structure of the game itself is developed.

CHAPTER IX

RATIONAL PROBABILITY MIXES

Advance knowledge cannot be gained from ghosts and spirits, inferred from phenomena or projected from measures of Heaven, but must be gained from men for it is the knowledge of the enemy's true situation.

SUN TZU, *ART OF WAR*

In finding a Bayesian Nash equilibrium to *Operation Overlord/Fortress Europe with Uncertainty* or the games of siege presented in the last two chapters, there was the need for prior knowledge of the probabilities with which specific courses of action would be chosen by the opposing player. In *Operation Overlord/Fortress Europe with Uncertainty*, Germany needs to know the probability with which Allies would choose the course of action to feign at Calais and attack at Normandy and the probability that the Allies would choose the course of action to attack Calais. It was in knowing that probability distribution that Germany was able to calculate the expected outcome valuations to select the most favorable course of action from within an information set. In the previous chapters, we presented these probability distributions as “historical precedent” or as “prior knowledge.” In this chapter, however, we will weaken that prior information requirement and allow those probability distributions to emerge as an equilibrium condition determined by the structure of the game.

First, the structure of simultaneous-play games will be used to develop the strategy for solving each player’s best probability distribution. When the players mutually select a preferred probability distribution, given their opponent’s chosen probability distribution, we will have solved for a set of equilibrium probability distributions. These are called Mixed Strategy Equilibria.¹

Having solved for a Mixed Strategy Equilibrium for a simultaneous-play game, how this solution technique can be applied to sequential-play games with information sets that can be represented in a simultaneous-play game format will be developed. From there *Operation Overlord/Fortress Europe with Uncertainty* will be expanded to include these mixed strategy equilibria.

I. Course-of-Action Mixes in Simultaneous-Play Games

In the simultaneous-play games discussed in Chapter III, the game play resolved to Nash equilibria where the players were each choosing specific courses of action. This is not always the case in discrete course-of-action simultaneous-play games. The best-known such game that fails to resolve to a discrete course-of-action equilibrium is the childhood choosing game Rock-Paper-Scissors. In games such as this one, the players resort to randomly choosing a course of action to keep their competition off guard and increase their potential outcome valuation. For the center in the hockey example from Chapter I, sometimes the head-move left is a fake, and sometimes it is him taking the puck left. Mixing up the course-of-action choice intuitively leaves the opponent off guard and has beneficial outcome possibilities for the player mixing it up.

Most people remember the game Rock-Paper-Scissors from their youth. This is a two-player simultaneous-play game. Each player can choose one of three courses of action: to be Rock, Paper, or Scissors. The players simultaneously reveal their course-of-action choice, after which the game resolves to outcomes as indicated in Figure 9.1. If the players choose the same course of action, then the outcome is a tie with an outcome valuation of zero. The course of action Rock is defeated by Paper, but Paper is defeated by Scissors, and Scissors are defeated by Rock. A win has an outcome valuation of one, while a loss has an outcome valuation of negative one.

This game is interesting because it has no discrete course-of-action Nash equilibrium. There is no intersection between course-of-action choices by each player, which fulfills the Nash equilibrium mutual-best-response criteria described in Chapter III. If the two players in the game, Player 1 and Player 2, both choose Rock and end up in the top left outcome grid of the game, they both have outcomes valued at zero and have a unilateral incentive to choose Paper to receive a higher outcome valuation. Player 1 would choose Paper given that

Player 2 had played Rock. Thus, Rock is not Player 1's best response to Player 2's choice of Rock. A similar situation applies to Player 2. Thus, the game play of Rock, then Rock again does not yield mutual-best responses. There is no outcome grid in this game for which one or both players do not have unilateral incentives to change their course-of-action choice.

Additionally, if it is known that Player 1 consistently chooses a specific course of action (Rock), then Player 2 will counter by consistently choosing the countering course of action (Paper). Thus, when playing this game, the players will "mix it up." They randomly choose one of their three available courses of action. As Avinash K. Dixit, Susan Skeath, and David H. Reiley Jr. show in *Games of Strategy*, the best mix is to choose each course of action randomly, such that each course of action has the equal probability of being chosen one third of the time.²

The Rock-Paper-Scissors Game

		Player 2		
		Rock	Paper	Scissors
Player 1	Rock	0 ↓ 1	-1 → 1	1 -1
	Paper	1 -1	0 0	-1 1
	Scissors	-1 1	1 -1	0 0

Figure 9.1: The game Rock-Paper-Scissors has no discrete course-of-action Nash equilibrium, but there is an equilibrium strategy to playing the game. In this game, the players should choose to "mix it up." That is, they should randomly choose a course of action with each of them probability-weighted at one-third.

Finding Mixed-Strategy Equilibria

To work through the process of finding Mixed-Strategy Equilibria to a simultaneous-play game, consider an adaptation to the normal form of a siege game first discussed in Chapter III and represented in Figure 3.2, then introduced again as a solution tool to an information set in Chapter V, Figure 6.6. There, the city had the course-of-action choice to close up its gates or to ambush the raiders. Finding themselves in Sub-game 2, the raiders had two course-of-action choices: to avoid the city or to lay siege to it. The course-of-action pairs, outcomes, and valuations to the city and the raiders were listed in Table 6.1 and represented in Table 9.1. The version of the simultaneous-play siege game will be modified in this chapter so there are no discrete course-of-action Nash equilibria.

Table 9.1: Choice and Outcome Pairs

	CHOICE PAIRS	OUTCOME PAIRS	RANK
1	C: Close Up R: Lay Siege	C: Pay Ransom R: No Glory & Ransom	C: 2 R: 1
2	C: Close Up R: Avoid City	C: Unmolested R: No Loot	C: 3 R: 2
3	C: Ambush R: Lay Siege	C: Destroyed and Looted R: Glory & Looting	C: 1 R: 4
4	C: Ambush R: Avoid City	C: Some Bravado R: Glory & No Loot	C: 4 R: 3

The normal form of this game is represented in Figure 9.2. This game has no discrete course-of-action equilibria, just as in the game Rock-Paper-Scissors. In every cell in the normal form of this game, one

of the two players could achieve a higher outcome valuation by unilaterally choosing a different course of action, given the opposing player's course-of-action choice. Consider the top left-hand cell of this game, in which the city chose the course of action to close its gates, and the raiders lay siege in response. In this cell, the city has an outcome valuation of two and the raiders have an outcome valuation of one. But, if the raiders knew the city was going to choose their course of action close, the raiders' best response would be to change their course-of-action choice from siege to avoid and increase their outcome valuation to $2 > 1$. The raiders' willingness to reevaluate their course-of-action choice is indicated in Figure 9.2 by the arrow pointing from the upper left-hand cell to the upper right-hand cell. This type of reasoning holds for every cell in the normal form of this game for at least one of the two players. None of the cells contain mutual best responses; no cell in Figure 9.2 has arrows from both players pointing into it. Thus, there is no discrete-strategy Nash equilibrium to the *Game of Siege with Ambush*.

Game of Siege with Ambush

		Raiders	
		Siege	Avoid
City	Close	2 ↑ 1	3 ↓ 2
	Ambush	1 ↓ 4	4 ← 3

Figure 9.2: This modified siege game has no discrete-strategy Nash equilibria. At least one player always can achieve a better outcome through a unilateral course-of-action change, as indicated by the arrows.

In this game, as in the game Rock-Paper-Scissors, if one player chooses to always use a particular course of action, their opponent will counter that course-of-action choice to the opponent's advantage. To prevent being taken advantage of, the players will choose to vary their course-of-action choice. Therefore, the city will choose to vary its course-of-action choice: to close with some probability ρ_{close} and to ambush the rest of the time with probability ρ_{ambush} . Since the city has only two course-of-action choices, to close or to ambush, then $\rho_{\text{close}} + \rho_{\text{ambush}} = 1$, and it must be that $\rho_{\text{ambush}} = 1 - \rho_{\text{close}}$.

The raiders must choose a course of action in response to the fact that the city has chosen to close, ρ_{close} . The raiders' course-of-action choice will be determined by the course of action that yields the highest expected outcome valuation given the city's choice of ρ_{close} . If the raiders choose to lay siege, their actual outcome valuations will be either one, if the city chooses the course of action to close, or four, if the city chooses the course of action to ambush. The raiders do not know exactly which course-of-action choice the city will make, but the raiders do know the probabilities across which they will be selected. So, their expected outcome valuation of their course-of-action choice to lay siege is

$$\text{EV}(\text{siege})_{\text{raiders}} = \rho_{\text{close}} \times 1 + \rho_{\text{ambush}} \times 4 = \rho_{\text{close}} \times 1 + (1 - \rho_{\text{close}}) \times 4.$$

Figure 9.3 calculates both of the expected valuations the raiders anticipate from both of their course-of-action choices, to lay siege to the city or to avoid the city. The expected valuation of the raiders' course-of-action choice siege is $\text{EV}(\text{siege})_{\text{raiders}} = 4 - \rho_{\text{close}} \times 3$.

This expected valuation will take on different levels, depending on the city's choice of ρ_{close} . For example, if the city chooses, at one extreme, $\rho_{\text{close}} = 1$ —that is, always choosing to close—the raiders can expect an outcome valuation of one for their responsive choice to lay siege. This makes sense, since the city and the raiders are choosing actions that result in the top right outcome cell in Figure 9.3. If, on the other hand, the city chooses to flip a coin to decide which course of action to choose, or $\rho_{\text{close}} = 0.50$, the raiders can expect an outcome valuation of

$$\text{EV}(\text{siege})_{\text{raiders}} = 4 - \rho_{\text{close}} \times 3 = 4 - 0.50 \times 3 = 2.50$$

for the course of action to lay siege to the city.

Raiders' Expected Valuations Given p_{close}

		Raiders	
		Siege	Avoid
City	p_{close} Close	2 1	3 2
	$1 - p_{\text{close}}$ Ambush	1 4	4 3

The raiders' expected outcome valuation for choosing siege given the city's probability to close p_{close} :

$$EV_{\text{raiders}} = p_{\text{close}} \times 1 + (1 - p_{\text{close}}) \times 4$$

or

$$EV_{\text{raiders}} = 4 - 3 \times p_{\text{close}}$$

The raiders' expected outcome valuation from choosing avoid given the city's probability to close p_{close} :

$$EV_{\text{raiders}} = p_{\text{close}} \times 2 + (1 - p_{\text{close}}) \times 3$$

or

$$EV_{\text{raiders}} = 3 - 2 \times p_{\text{close}}$$

Figure 9.3: The city chooses the probability p_{close} , such that the raiders are indifferent to the choice to siege or to avoid.

At the other extreme, if the city always chooses the course of action to ambush ($p_{\text{close}} = 0$), then the game resolves to the bottom right outcome valuation cell, and the raiders' outcome valuation is four. Consequently, the raiders' expected valuation for choosing the course of action to lay siege ranges from a high of four to a low of one, as the city's choice of p_{close} changes from 0 to 1.

Using exactly the same thought process employed to determine the expected valuations to the raiders for the course-of-action choice to lay siege, given the city's probability choice p_{close} , the expected valuation to the raiders of the course-of-action choice to avoid the city is calculated as $EV(\text{avoid})_{\text{raiders}} = 3 - p_{\text{close}}$. Therefore, the raiders' expected valuation for choosing the course of action avoid ranges from a high of three to a low of two, as the city's choice of p_{close} changes from zero to one.

In Figure 9.4, the raiders' potential expected valuations, given the city's choice of ρ_{close} , is illustrated for the raiders' course-of-action choices both to lay siege and avoid the city. If the city randomly chooses the course of action to close, with a frequency ranging from never to less than half of the time ($0 \leq \rho_{\text{close}} < \frac{1}{2}$), the raiders' expected valuation for choosing to lay siege is greater than their expected valuation of choosing to avoid. Thus, if the raiders know that $0 \leq \rho_{\text{close}} < \frac{1}{2}$, then the raiders would always choose to lay siege, and their best response to $0 \leq \rho_{\text{close}} < \frac{1}{2}$ is to choose the course of action to lay siege. If the city randomly chooses the course of action to close, with a frequency ranging from more than half the time to always ($\frac{1}{2} < \rho_{\text{close}} \leq 1$), the raiders' expected valuation for choosing to avoid is greater than the expected valuation of choosing to lay siege. Thus, if the raiders know that $\frac{1}{2} < \rho_{\text{close}} \leq 1$, then their best response is always to choose siege. If the city flips a coin and randomly chooses to close exactly half of the time ($\rho_{\text{close}} = \frac{1}{2}$), the raiders' expected valuation for choosing either siege or avoid is the same, and they should be indifferent to the course of action siege or avoid.

Given the raiders' two expected valuations for their dueling course-of-action choices, the raiders will choose the course of action that yields the largest expected valuation given the city's probability choice, ρ_{close} . The city, however, does not want to select a probability choice ρ_{close} that will give any advantage to the raiders. The city will choose the probability ρ_{close} that leaves the raiders indifferent to their choice to lay or to avoid the city entirely. This means the raiders' expected valuation for laying siege is equal to their expected valuation for avoiding. This means

$$EV(\text{siege})_{\text{raiders}} = 4 - \rho_{\text{close}} \times 3 = 3 - \rho_{\text{close}} = EV(\text{avoid})_{\text{raiders}},$$

so $1 = \rho_{\text{close}} \times 2$ or $\rho_{\text{close}} = \frac{1}{2}$.

The city's most advantageous probability choice, given the raiders' propensity to select a course of action that will exploit the city's choice (to siege or to avoid), is to choose the probability to close its gates, $\rho_{\text{close}} = \frac{1}{2}$.

Raiders' Expected Valuations Given p_{close}

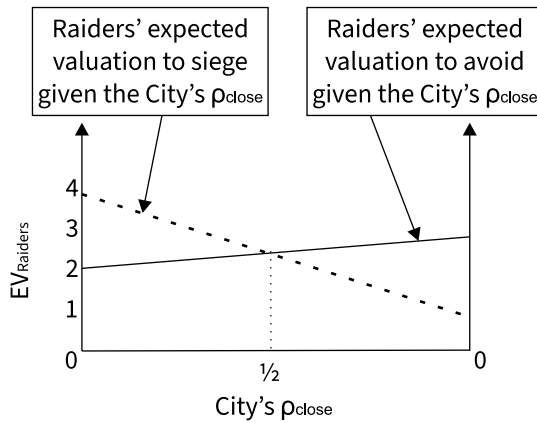


Figure 9.4: If the city chooses $p_{\text{close}} < 1/2$, then the raiders' expected valuation for the course-of-action choice siege is higher than their expected valuation for the course-of-action choice avoid. If the city chooses $p_{\text{close}} > 1/2$, then the raiders' expected valuation for the course-of-action choice avoid is higher than their expected valuation for the course-of-action choice siege. If the city chooses $p_{\text{close}} = 1/2$, then the raiders' expected valuation for the course-of-action choice siege is the same as their expected valuation for the course-of-action choice avoid.

Now consider the problem of a city facing an onslaught of raiders. The raiders know that if they choose only one of their two available courses of action, to siege or to avoid, the city will choose a course of action to exploit the raiders' predictability. This means that the raiders will choose to mix up their course-of-action choice as well. They will choose the course of action to lay siege with some probability ϕ_{siege} and the course of action to avoid the city the rest of the time with probability ϕ_{avoid} . Since the raiders only have two course-of-action choices, to siege or to avoid, then $\phi_{\text{siege}} + \phi_{\text{avoid}} = 1$, and $\phi_{\text{avoid}} = 1 - \phi_{\text{siege}}$.

Using the same analytical process that generated the raiders' expected valuations, the city has an expected valuation for choosing the course of action to close of

$$EV(\text{close})_{\text{city}} = \phi_{\text{siege}} \times 2 + (1 - \phi_{\text{siege}}) \times 3 = 3 - \phi_{\text{siege}},$$

and an expected valuation for choosing the course of action ambush of $EV(\text{ambush})_{\text{city}} = 4 - 3 \times \phi_{\text{siege}}$, as presented in Figure 9.5.

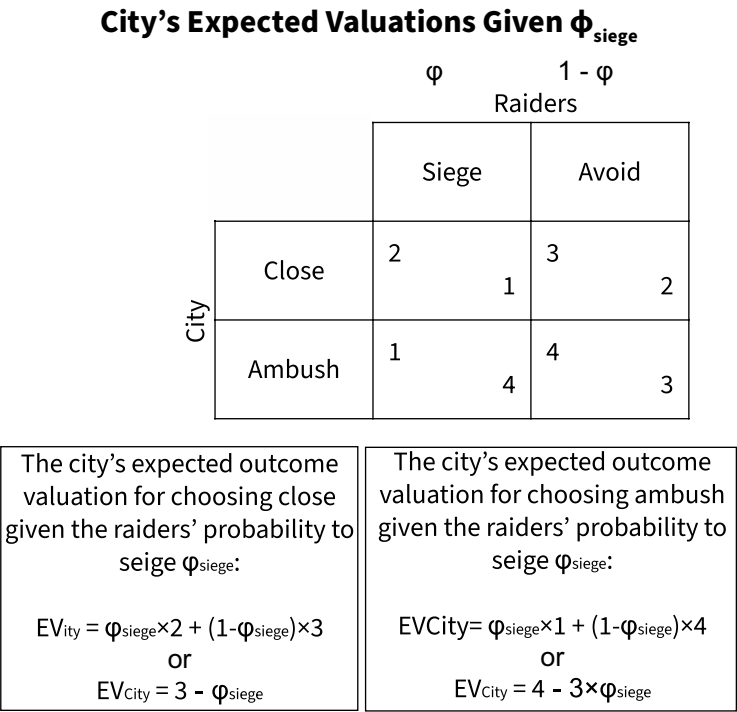


Figure 9.5: The raiders choose the probability ϕ_{siege} , such that the city is indifferent to the choice to close or to ambush.

Given the city's expected valuations from their two course-of-action choices—to close its gates or to ambush the raiders—the city will choose whatever course of action yields the largest benefit, given the raiders' probability choice ϕ_{siege} . The raiders, however, do not want to choose to ϕ_{siege} , because that will give an advantage to the city. The raiders will choose the probability ϕ_{siege} , because that leaves the city indifferent to choosing between their course of action to close and their course of action to ambush. This means the city's expected valuation for closing its gates is equal to its expected valuation for ambushing the raiders. Therefore,

$$\text{EV}(\text{close})_{\text{city}} = 3 - \phi_{\text{siege}} = 4 - 3 \times \phi_{\text{siege}} = \text{EV}(\text{ambush})_{\text{city}},$$

so, $1 = \phi_{\text{siege}} \times 2$ or $\phi_{\text{siege}} = \frac{1}{2}$. Given the city's propensity to choose from its course-of-action choices (to close or to ambush) to exploit the raiders, the raiders' most advantageous probability choice in turn is to choose to lay siege to the city, $\phi_{\text{siege}} = \frac{1}{2}$.

Again, as with the discussion of the raiders' expected valuations, the expected valuations for the city's decision to close or to ambush, given the raiders' probability choice ϕ_{siege} , are presented in Figure 9.6. If the city knows that $0 \leq \phi_{\text{siege}} < \frac{1}{2}$, then the city's best response is to always choose to ambush. If the city knows that $\frac{1}{2} < \phi_{\text{siege}} \leq 1$, then the city's best response is to always choose to close its gates. Finally, if $\phi_{\text{siege}} = \frac{1}{2}$, then the city is indifferent to choosing a course of action between the two.

City's Expected Valuations to Raiders' ϕ_{siege}

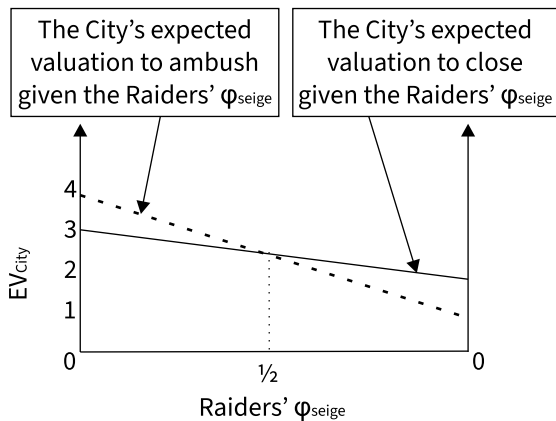


Figure 9.6: If the raiders choose $\phi_{\text{siege}} < \frac{1}{2}$, then the city's expected valuation for the course-of-action choice ambush is higher than its expected valuation for the course-of-action choice close. If the city chooses $\phi_{\text{siege}} > \frac{1}{2}$, then the city's expected valuation for the course-of-action choice close is higher than its expected valuation for the course-of-action choice ambush. If the raiders choose $\phi_{\text{siege}} = \frac{1}{2}$, then the city's expected valuation for the course-of-action choice close is the same as its expected valuation for the course-of-action choice ambush.

Given the city’s expected valuations represented in Figure 9.6, the raiders’ best response to the city choosing to ambush ($\rho = 0$) is to always siege ($\phi = 1$). In fact, if the city chooses $\rho < \frac{1}{2}$, the raiders should choose $\phi = 1$, or always lay siege. The raiders’ best response to the city choosing $\rho < \frac{1}{2}$ is to choose $\phi = 1$. Similarly, the raiders’ best response to the city choosing $\rho > \frac{1}{2}$ is to choose $\phi = 0$, or always to avoid the city. Finally, the raiders’ best response to the city choosing $\rho = \frac{1}{2}$ is to choose $0 \leq \phi \leq 1$, or ϕ can be anything. The raiders’ best-response function can be written as illustrated in Equation 9.1 and graphed as in Figure 9.7.

Equation 9.1: Raiders’ Best Response Function

$$\phi = f(\rho) = \begin{cases} 0, & \text{if } \rho > \frac{1}{2}, \\ 0 \leq \phi \leq 1, & \text{if } \rho = \frac{1}{2}, \\ 1, & \text{if } \rho < \frac{1}{2} \end{cases}$$

Raiders’ Best Response to City’s ρ Choice

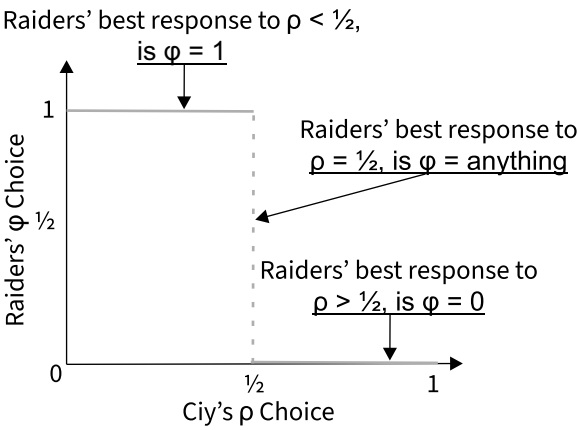


Figure 9.7

The logic applied to generate the raiders' best-response function can be applied to generate the city's best-response function as well. Given the city's expected valuations for its course-of-action choices, to close or to ambush, given the raiders' probability choice of ϕ , represented in Figure 9.5, the city's best response to the raiders choosing to siege ($\phi = 1$) is to always close ($\rho = 1$). In fact, if the raiders choose $\phi > \frac{1}{2}$, the city should choose $\rho = 1$, or always close. The city's best response to the raiders choosing $\phi > \frac{1}{2}$ is to choose $\rho = 1$. Similarly, the city's best response to the raiders choosing $\phi < \frac{1}{2}$ is to choose $\rho = 0$, always ambush. Finally, the city's best response to the raiders choosing $\phi = \frac{1}{2}$ is to choose $0 \leq \rho \leq 1$, or ρ , which can be anything. The city's best-response function can be written as shown in Equation 9.2 and graphed as in Figure 9.8.

Equation 9.2: Title City's Best Response Function

$$\rho = f(\phi) = \begin{cases} 0, & \text{if } \phi < \frac{1}{2}, \\ 0 \leq \rho \leq 1, & \text{if } \phi = \frac{1}{2}, \\ 1, & \text{if } \phi > \frac{1}{2} \end{cases}$$

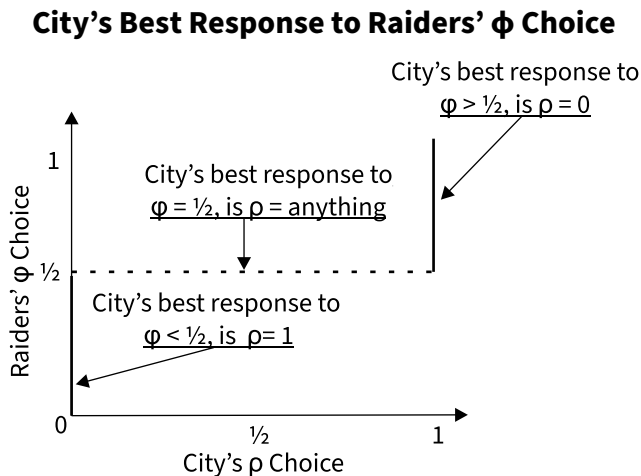


Figure 9.8

Given the two best-response relationships for the city and the raiders as written in Equations 9.1 and 9.2 and represented in Figures 9.7 and 9.8, there is a set of probability choices— ρ_{close} by the city and ϕ_{siege} by the raiders—which satisfies the Nash equilibrium condition set forth in Chapter III: when individual players cannot achieve a higher-valued outcome through unilateral course-of-action changes.

This condition is met when the players' course-of-action choices mutually fulfill each of their best-response conditions. In this case, that would require the city's best-response probability choice of ρ_{close} , given the raiders' probability choice ϕ_{siege} , to elicit the same ϕ_{siege} as the raiders' best-response probability, given the city's best-response probability ρ_{close} .

The mutually fulfilled best-response criterion for the mixed strategy equilibrium for this modified siege game is best illustrated in Figure 9.9, where the best-response functions of both players intersect. The intersection of the best-response functions represents the condition where the probability-choice pair $(\rho_{\text{close}}, \phi_{\text{siege}})$ result in both the city's and the raiders' highest expected valuation, given the other's probability choice. In this game, a mixed-strategy equilibrium is achieved when the city chooses to mix up its course-of-action choice to close or to ambush by choosing randomly to close, with a probability of $\rho_{\text{close}} = \frac{1}{2}$. Similarly, equilibrium is achieved when the raiders choose to mix up their course-of-action choice to siege or to avoid the city by choosing randomly to siege, with a probability $\phi_{\text{siege}} = \frac{1}{2}$.

When playing this mixed-strategy equilibrium, the city has an expected outcome valuation of

$$\begin{aligned} EV_{\text{city}} &= 3 - \phi_{\text{siege}} = 3 - (\frac{1}{2}) = 2.5, \text{ or} \\ EV_{\text{city}} &= 4 - 3 \times \phi_{\text{siege}} = 4 - 3 \times (\frac{1}{2}) = 2.5. \end{aligned}$$

And the raiders have an expected outcome valuation of

$$\begin{aligned} EV_{\text{raiders}} &= 4 - 3 \times \rho_{\text{close}} = 4 - 3 \times (\frac{1}{2}) = 2.5, \text{ or} \\ EV_{\text{raiders}} &= 3 - \rho_{\text{close}} = 3 - (\frac{1}{2}) = 2.5. \end{aligned}$$

This idea of mixed-strategy equilibria developed in the *Game of Siege with Ambush* just presented gives a framework for discerning

each player's best-probability mix given their opponents' propensity to vary their own course-of-action choice. In the case of the raiders, the raiders can deduce the best-probability mix that the city will use across its course-of-action choices to close and ambush. In Chapters V and VI, the raiders relied on information that was external to the actual structure of the game for the probability that the city had chosen the course of action to close or the course of action to ambush. That probability distribution was either historically determined or was genetically encoded with outside knowledge of the distribution of genotypes.

In the game presented in Chapter VIII, *Operation Overlord/Fortress Europe with Uncertainty*, Germany relied on external "knowledge" of the Allies' propensity to choose between the course of action to feign at Normandy and attack Calais and the course of action to attack Normandy. Germany "knew" the Allies' probability distribution was $\rho_{\text{feign Normandy}} = 0.70$ and $\rho_{\text{attack Normandy}} = 0.30$.

Mixed-Strategy Nash Equilibrium

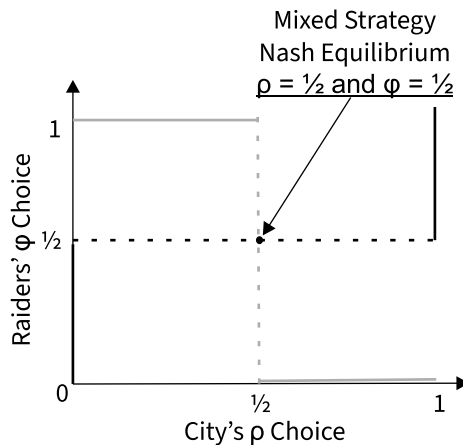


Figure 9.9: The mixed-strategy Nash equilibrium for the simultaneous-play *Game of Siege with Ambush* occurs where the best-response functions of both players, the city and the raiders, intersect. At the point $\rho = \frac{1}{2}$ and $\phi = \frac{1}{2}$, neither the city nor the raiders have any incentive to change their mix of close/ambush or siege/avoid, respectively.

It is reasonable to ask whether Germany's "knowledge" of the Allies' probability distribution made sense given the structure of the game. The next section of this chapter will use the mixed-strategy equilibrium technique to find the equilibrium-probability distributions, given the game structure. This technique makes these probability distributions endogenous to the situation being analyzed by Germany's and the Allies' strategists.

II. Mixed Strategies in *Operation Overlord/Fortress Europe*

In the sequential-play game *Operation Overlord/Fortress Europe with Uncertainty* presented in Chapter VIII, Germany had the first decision node with six possible courses of action. The three course-of-action choices where Germany holds its mobile forces inland as reserves are the relevant ones for this discussion. It is Germany choosing to hold its mobile forces inland in reserve that gives the Allies the opportunity to engage in the deceptive tactics that generate information sets that have more than one German decision node. In Chapter VIII, we also resolved Germany's information set problem using Bayesian equilibrium techniques. Germany needed to know the probability distribution across their decision nodes in the information set to calculate expected valuations for the course-of-action choices. In Chapter VIII, the probability distribution was assumed to be known by Germany. In this section, Germany and the Allies are going to come to a mixed-strategy equilibrium. There is no longer a probability distribution that is *a priori* known; it will be determined by the game's structure.

As presented in Chapter VIII's Figure 8.4, reproduced here as Figure 9.10, when Germany chooses the course of action to build Calais mobile inland, the Allies have four course-of-action choices: to attack Normandy; to feign at Normandy and attack Calais; to feign at Calais and attack at Normandy; or to attack Calais. The Germans cannot distinguish between the Allied course-of-action choices to attack at Normandy and to feign at Normandy and attack Calais, producing

Information Set 1 in Figure 9.10. Similarly, Information Set 2 exists because the Allies' courses of action—to feign at Calais and attack Normandy, and to attack at Calais—are observationally equivalent to Germany's. In Chapter VIII, to resolve this game up one more level, Germany needed to know the probability distribution across the decision nodes in each of the two information sets.

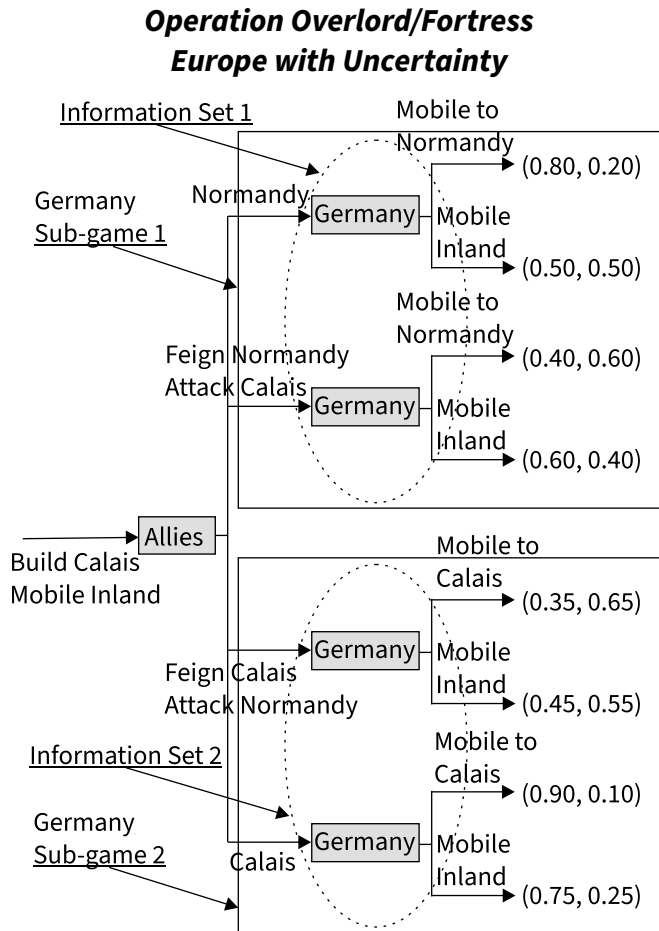


Figure 9.10: In this branch of the game, the Allied and German course-of-action pairs are linked to their respective expected outcome valuations (EV_{Germany} , EV_{Allies}). At this point in the game's solution, Germany has two sub-games, each coincidental with its two information sets.

In Chapter VI, we showed that a sub-game with two decision nodes inside an information set could be viewed as a simultaneous-play game, because neither player can know what course of action the opposing player has chosen prior to their course-of-action choice. Hence, even if the choices are sequential in time, they are not sequential in information. Thus, these two sub-games in Figure 9.10, represented in extensive-play form, can be represented as simultaneous-play games in normal form.

Sub-game 1 from Figure 9.10 is converted to its normal form equivalent in Figure 9.11. This simultaneous-play form of Sub-game 1 has no Nash equilibrium in its discrete course-of-action choices. In every cell of the normal-form representation of this sub-game, either Germany or the Allies have a unilateral incentive to choose a different course of action. For example, if the Allies choose to attack Normandy and Germany chooses to mobile to Normandy, the game resolves to the Allies gaining an outcome valuation of 0.2 and Germany gaining an outcome valuation of 0.8, as shown in the upper left-hand cell of Figure 9.11. In this situation, the Allies have the unilateral incentive to change their course-of-action choice from attacking Normandy to feigning at Normandy and attacking Calais, increasing their outcome valuation from the current 0.2 to the possible 0.6.

The conditions in Sub-game 1 of *Operation Overlord/Fortress Europe with Uncertainty* that contribute to the sub-game not having a discrete course-of-action Nash equilibrium means that if Germany or the Allies consistently choose a course of action, the other will exploit that consistency to their advantage. Thus, if Germany consistently chooses to mobile to Normandy, the Allies will exploit this consistent German behavior and choose to feign at Normandy and attack Calais, which would be to Germany's disadvantage and the Allies' benefit. Given both Germany's and the Allies' exploitive behavior, it is in the best interest of both to mix up their course-of-action choices. The Allies will choose a probability distribution across their course-of-action choices by choosing to attack at Normandy, with probability ρ_{AN} , and to feign at Normandy and attack Calais, with probability ρ_{FN} , where $\rho_{AN} + \rho_{FN} = 1$, or $\rho_{FN} = 1 - \rho_{AN}$.

Operation Overlord/Fortress Europe with Uncertainty

		Germany	
		Sub-game 1	
Allies		Mobile to Normandy	Mobile Inland
	Attack Normandy	0.2 ↓	0.5 ↓ 0.5 ←
	Feign Normandy Attack Calais	0.6 ↓ 0.4 →	0.4 ↓ 0.6 →

Figure 9.11: Because Germany's information set does not allow its leaders to know whether the Allies have chosen the course of action attack Normandy or the course of action feign Normandy attack Calais, Sub-game 1 can be represented as this simultaneous-play game. This game, as indicated by the arrows, has no discrete course-of-action Nash equilibrium.

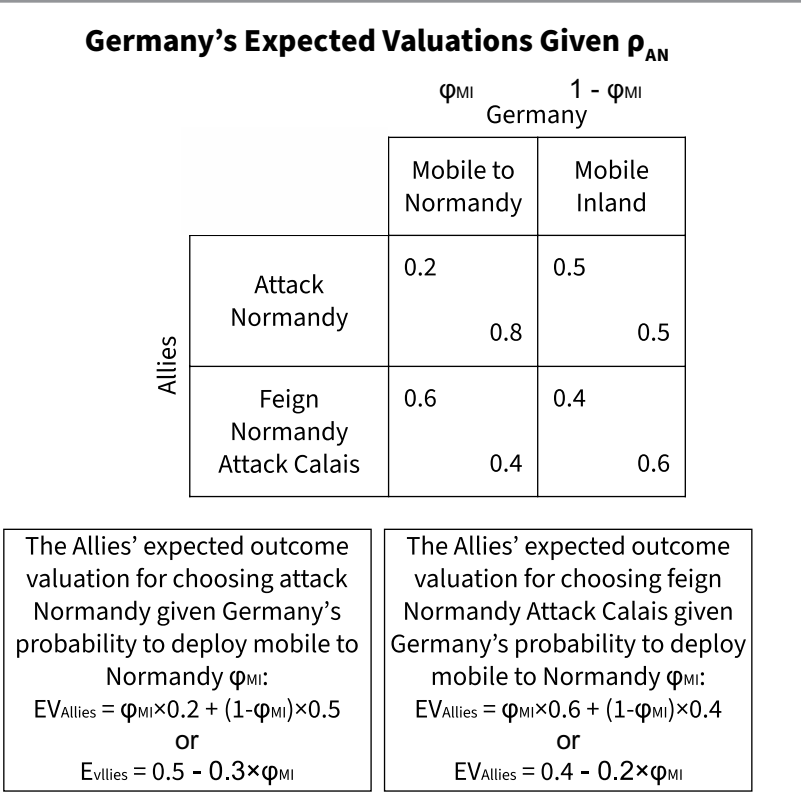
Germany must choose a course of action given the probability that the Allies have chosen to attack at Normandy, ρ_{AN} . Germany's course-of-action choice will be determined by the course of action that yields the highest expected outcome valuation, given the Allies' choice of ρ_{AN} . Germany does not know exactly which course-of-action choice the Allies will make, but the Germans do know that there are probabilities from which the allies will choose. Therefore, Germany's expected outcome valuation of the course-of-action choice to deploy its mobile forces to Normandy is

$$EV(\text{Mobile to Normandy})_{\text{Germany}} = \rho_{AN} \times 0.80 + \rho_{FN} \times 0.40 = \rho_{AN} \times 0.80 + (1 - \rho_{AN}) \times 0.40.$$

$EV(\text{Mobile to Normandy})_{\text{Germany}} = 0.40 + \rho_{AN} \times 0.40$. Similarly, Germany's expected valuation of its course-of-action choice to hold its mobile forces inland, given the Allies' choice of ρ_{AN} , is

$$EV(\text{Mobile Inland})_{\text{Germany}} = 0.60 - \rho_{AN} \times 0.10.$$

Germany’s expected valuations for each of its course-of-action choices—to deploy mobile forces to Normandy and to hold mobile forces inland—are developed and illustrated in Figure 9.12.



course of action to attack at Normandy moves from always attacking at Normandy ($p_{AN} = 1$) to always feigning at Normandy and attacking Calais ($p_{AN} = 0$), Germany's expected valuation of its course-of-action choice to hold mobile inland varies from 0.50 to 0.60, also indicated in Figure 9.13.

The Allies will select the probability that they attack at Normandy because the expected outcome valuation for Germany same for either Germany's course-of-action choice to deploy mobile forces to Normandy or to hold mobile forces inland. Thus,

$$\begin{aligned} EV(\text{Mobile to Normandy})_{\text{Germany}} &= 0.40 + p_{AN} \times 0.40 = \\ 0.60 - p_{AN} \times 0.10 &= EV(\text{Mobile Inland})_{\text{Germany}}, \text{ or} \\ p_{AN} \times 0.50 &= 0.20, \text{ so } p_{AN} = 0.20/0.50 = 2/5, \end{aligned}$$

as indicated in Figure 9.13.

Germany's Expected Valuations Given p_{AN}

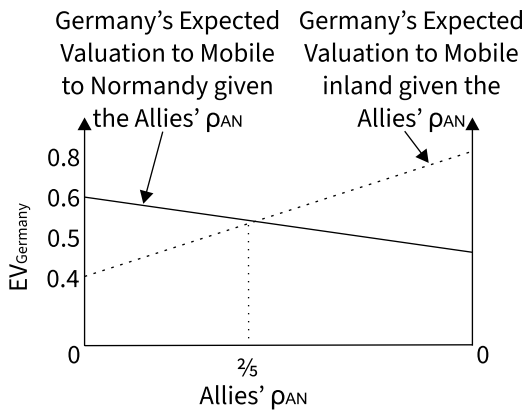


Figure 9.13: If the Allies choose $p_{AN} < 2/5$, then Germany's expected outcome valuation for the course-of-action choice mobile to Normandy is higher than its expected outcome valuation for the course-of-action choice mobile inland. If the Allies choose $p_{AN} > 2/5$, then Germany's expected outcome valuation for the course-of-action choice mobile inland is higher than its expected outcome valuation for the course-of-action choice mobile to Normandy. If the Allies choose $p_{AN} = 2/5$, then Germany's expected valuation for the course-of-action choice mobile to Normandy is the same as its expected valuation for the course-of-action choice mobile inland.

Germany, like the Allies, will avoid its opponent's exploitive course-of-action choices in response to predictable German course-of-action choices by randomly mixing up its own choices. Thus, Germany will choose to deploy mobile forces to Normandy, with probability ϕ_{MN} , or to hold mobile forces inland, with probability ϕ_{MI} ; thus, $\phi_{AN} + \phi_{FN} = 1$, or $\phi_{FN} = 1 - \phi_{AN}$.

Since Germany is mixing its course-of-action choice with probability ϕ_{MN} , the Allies will maximize their expected valuation, given Germany's probability distribution ϕ_{MN} . The Allies do not know exactly which course-of-action choice Germany will make, but the Allies do know there are probabilities across which Germany will select each course of action. Therefore, the Allies' expected outcome valuation of the course-of-action choice to attack at Normandy is

$$\begin{aligned} EV(\text{Attack Normandy})_{\text{Allies}} &= \phi_{MN} \times 0.20 + \phi_{MI} \times 0.50 = \\ &\phi_{MN} \times 0.20 + (1 - \phi_{MI}) \times 0.50. \end{aligned}$$

$EV(\text{Attack Normandy})_{\text{Allies}} = 0.50 - \phi_{MN} \times 0.30$. Similarly, the Allies' expected valuation for their course-of-action choice to feign at Normandy and attack Calais, given Germany's choice of ϕ_{MN} , is

$$EV(\text{Feign Normandy Attack Calais})_{\text{Allies}} = 0.40 + \phi_{MN} \times 0.20.$$

The Allies' expected valuations for their course-of-action choices to either attack at Normandy or feign to Normandy and attack Calais are developed and illustrated in Figure 9.14.

Germany will select the probability that it chooses the course of action mobile to Normandy, such that the expected outcome valuation for the Allies is the same for the Allies' course-of-action choice attack Normandy and their course-of-action choice to feign Normandy and attack Calais. Thus,

$$\begin{aligned} EV(\text{Attack Normandy})_{\text{Allies}} &= 0.50 - \phi_{MN} \times 0.30 = \\ 0.40 + \phi_{MN} \times 0.20 &= EV(\text{Feign Normandy Attack Calais})_{\text{Allies}}, \end{aligned}$$

or $\phi_{MN} \times 0.50 = 0.10$, so $\phi_{MN} = 0.10/0.50 = 1/5$, as indicated in Figure 9.15.

Given the Germans' expected valuations represented in Figure 9.13, their best response to the Allies choosing the course of action to feign at Normandy and attack Calais ($\rho = 0$) is to always choose to hold mobile forces inland ($\phi = 0$). In fact, if the Allies choose $\rho < 2/5$, the Germans should choose $\phi = 0$. Germany's best response to the Allies choosing $\rho < 2/5$ is to choose $\phi = 0$. Similarly, Germany's best response to the Allies choosing $\rho > 2/5$ is to choose $\phi = 1$. Finally, Germany's best response to the Allies choosing $\rho = 2/5$ is to choose $0 \leq \phi \leq 1$, or ϕ can be anything. Germany's best-response function can be written as illustrated in Equation 9.3 and graphed as in Figure 9.16.

Germany Chooses Its Probability ϕ_{MI}

		ϕ_{MI} 1 - ϕ_{MI} Germany	
		Mobile to Normandy	Mobile Inland
Allies	Attack Normandy	0.2 0.8	0.5 0.5
	Feign Normandy Attack Calais	0.6 0.4	0.4 0.6

The Allies' expected outcome valuation for choosing attack Normandy given Germany's probability to deploy mobile to Normandy ϕ_{MI} :

$$EV_{Allies} = \phi_{MI} \times 0.2 + (1 - \phi_{MI}) \times 0.5$$

or

$$EV_{Allies} = 0.5 - 0.3 \times \phi_{MI}$$

The Allies' expected outcome valuation for choosing feign Normandy Attack Calais given Germany's probability to deploy mobile to Normandy ϕ_{MI} :

$$EV_{Allies} = \phi_{MI} \times 0.6 + (1 - \phi_{MI}) \times 0.4$$

or

$$EV_{Allies} = 0.4 - 0.2 \times \phi_{MI}$$

Figure 9.14: Germany chooses the probability ϕ_{MI} , such that the Allies are indifferent to the choice to attack at Normandy or to feign to attack Normandy and attack at Calais.

Allies' Expected Valuations Given ϕ_{MI}

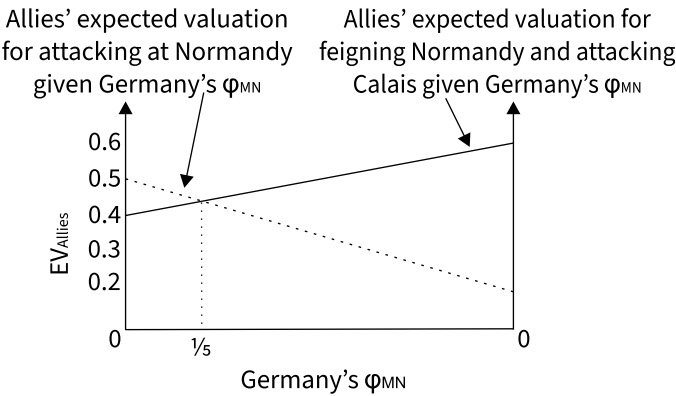


Figure 9.15: If Germany chooses $\phi_{MN} < 1/5$, then the Allies' expected outcome valuation for the course-of-action choice attack Normandy is higher than their expected outcome valuation for the course-of-action choice feign Normandy attack Calais. If Germany chooses $\phi_{MN} > 1/5$, then the Allies' expected outcome valuation for the course-of-action choice feign Normandy attack Calais is higher than their expected outcome valuation for the course-of-action choice attack Normandy. If Germany chooses $\phi_{MN} = 1/5$, then the Allies' expected valuation for the course-of-action choice attack Calais is the same as their expected valuation for the course-of-action choice feign Normandy attack Calais.

Equation 9.3: Germany's Best-Response Function

$$\varphi = f(\rho) = \begin{cases} 0, & \text{if } \rho < \frac{1}{5}, \\ 0 \leq \varphi \leq 1, & \text{if } \rho = \frac{1}{5}, \\ 1, & \text{if } \rho > \frac{1}{5}. \end{cases}$$

Using the same thought process, with the Allies' expected outcome valuations in Figure 9.15, the Allies' best response to Germany always choosing the course of action to hold mobile forces inland ($\phi = 0$) is to always choose to attack Normandy ($\rho = 1$). Whenever Germany chooses $\phi < 1/5$, the Allies' best response to Germany's probability choice is to choose the probability $\rho = 1$. Similarly, if Germany chooses

$\phi > 1/5$, the Allies' best response to Germany's probability choice is to choose the probability $\rho = 0$. Finally, if Germany chooses $\phi = 1/5$, the Allies' best response to Germany's probability choice is to choose the probability $0 \leq \rho \leq 1$, or ρ can be anything. The Allies' best-response function can be written as illustrated in Equation 9.4 and graphed as in Figure 9.16.

Mixed-Strategy Nash Equilibrium

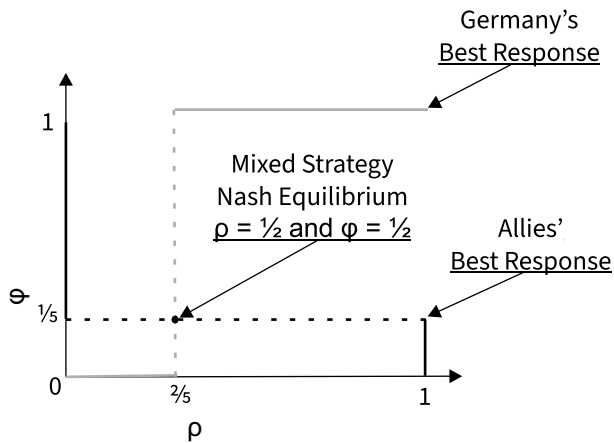


Figure 9.16: The Allies' and Germany's best-response functions are graphically depicted. The mixed-strategy Nash equilibrium for the simultaneous-play equivalent for Sub-game 1 occurs where the two best-response functions intersect—that is, where the Allies choose $\rho = 2/5$ and Germany chooses $\phi = 1/5$. Neither Germany nor the Allies have any incentive to change their respective probability choices.

Equation 9.4: Allies' Best-Response Function

$$\rho = f(\phi) = \begin{cases} 0, & \text{if } \phi > 1/5, \\ 0 \leq \rho \leq 1, & \text{if } \phi = 1/5, \\ 1, & \text{if } \phi < 1/5. \end{cases}$$

At this mixed-strategy equilibrium ($\phi = 1/5$, $\rho = 2/5$), Germany's expected outcome valuation in Sub-game 1 is

$$EV(\text{Mobile to Normandy})_{\text{Germany}} = 0.8 \times 2/5 + 0.4 \times 3/5 = 0.56$$

and

$$EV(\text{Mobile Inland})_{\text{Germany}} = 0.5 \times 2/5 + 0.6 \times 3/5 = 0.56,$$

regardless of the course-of-action choice. Similarly, the Allies' expected outcome valuation in Sub-game 1 is

$$EV(\text{Attack Normandy})_{\text{Allies}} = 0.2 \times 1/5 + 0.5 \times 4/5 = 0.44$$

and

$$EV(\text{feign Normandy attack Calais})_{\text{Allies}} = 0.6 \times 1/5 + 0.4 \times 4/5 = 0.44,$$

regardless of their course-of-action choice. Thus, Sub-game 1 of *Operation Overlord/Fortress Europe with Uncertainty* resolves to the mixed-strategy equilibrium in which the Allies choose to attack at Normandy with the probability of $\rho = 2/5$ and an expected outcome valuation of $EV_{\text{Allies}} = 0.44$. Using the same process, Germany chooses to deploy mobile forces to Normandy with the probability of $\phi = 1/5$ and an expected outcome valuation of $EV_{\text{Germany}} = 0.56$.

Having found the mixed-strategy equilibrium to Sub-game 1, our focus moves to Sub-game 2, represented in the normal form of the simultaneous-play equivalent game shown in Figure 9.17. This is possible because the same information reasons that sub-game 1 could be represented that way.

Inspecting the normal-form representation of Sub-game 2 allows us to immediately see that the Allies have a dominant course of action to feign at Calais and instead attack Normandy. That is, regardless of Germany's course-of-action choice, the Allies' outcome valuation is always higher when they choose to feign at Calais and attack at Normandy. Thus, in Sub-game 2, the Allies will always choose as their course of action to feign at Calais and attack at Normandy. In

response, Germany—knowing that in Sub-game 2 the Allies will always choose this as their course of action—will always respond by holding mobile forces inland. If we let p be the probability that the Allies will choose the course of action to attack at Calais and let ϕ represent the probability that Germany will choose the course of action to deploy mobile forces to Calais, then the Allies will choose $p = 0$ and Germany will choose $\phi = 0$, yielding an outcome valuation of 0.55 to the Allies and an outcome valuation of 0.45 to Germany in Sub-game 2, as shown in Figure 9.17.

**Operation Overlord/Fortress Europe
with Uncertainty**

		Germany	
Sub-game 2		Mobile to Normandy	Mobile Inland
Allies	Attack Normandy	0.10 ↓	0.25 ↓
	Feign Normandy Attack Calais	0.65 ↓	0.55 ↓
		0.90 ←	0.75 ←
		0.35 →	0.45 →

Nash
Equilibrium

Figure 9.17: Because Germany's information set does not allow it to know whether the Allies have chosen the course of action attack Calais or the course of action feign Calais attack Normandy, Sub-game 2 can be represented as this simultaneous-play game. This game, as indicated by the arrows, has a discrete course-of-action Nash equilibrium in which the Allies choose a dominant course-of-action choice feign Calais attack Normandy and Germany chooses its course-of-action choice mobile inland.

Having solved the outcomes for Sub-game 1 and Sub-game 2, those outcomes can be substituted into the branch of the game *Operation Overlord/Fortress Europe with Uncertainty* as introduced in Figure 9.10 and updated in Figure 9.18. Resolving up this branch of the game, the Allies have a higher equilibrium expected outcome valuation of $EV_{\text{Allies}} = 0.45$ by choosing to be in Sub-game 2, than their expected outcome valuation of $EV_{\text{Allies}} = 0.44$ by choosing to be in the mixed-strategy equilibrium in Sub-game 1. Therefore, in this branch of the game, the Allies will choose to feign at Calais and attack Normandy, and Germany in response will choose its course of action to hold its mobile forces inland, with expected outcome valuations of $EV_{\text{Allies}} = 0.45$ and $EV_{\text{Germany}} = 0.55$.

There are two more branches of the game *Operation Overlord/Fortress Europe with Uncertainty* that need to be treated in a way similar to the branch where Germany chooses its course of action to build at Calais and hold mobile forces inland. These two branches, highlighted in Figure 8.7, are when Germany chooses the course of action to build evenly sized fixed coastal defenses while holding mobile forces inland, as well as the course of action to build fixed coastal defenses at Normandy with mobile forces inland.

Starting with the branch of the game *Operation Overlord/Fortress Europe with Uncertainty* initiated by Germany choosing the course of action to build evenly sized mobile forces inland, this branch is replicated from Figure 8.5 as Figure 9.19, and we have Sub-game 3 and Sub-game 4 to resolve.

Translating the extensive form of the sequential-play game in Sub-game 3 from Figure 9.19 into its equivalent normal-form simultaneous-play game, we get Figure 9.20. This game, as indicated by the arrows, has a discrete course-of-action Nash equilibrium in which the Allies choose the dominant course of action feign Normandy attack Calais and Germany chooses its course of action mobile inland, yielding an outcome valuation of 0.55 to the Allies and an outcome valuation of 0.45 to Germany in Sub-game 3, as shown in Figure 9.20.

Operation Overlord/Fortress Europe with Uncertainty

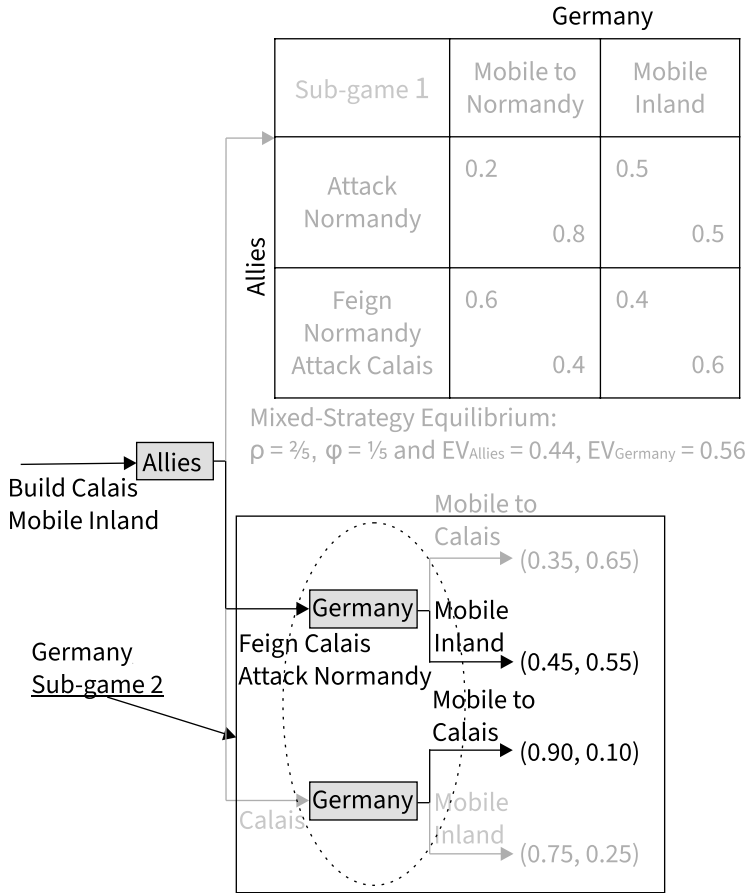


Figure 9.18: Resolving up this branch of the game, the Allies have a higher equilibrium-expected outcome valuation of $EV_{Allies} = 0.45$ by playing in Sub-game 2 than their expected outcome valuation of $EV_{Allies} = 0.44$ in the mixed-strategy equilibrium in Sub-game 1. Therefore, in this branch of the game, the Allies will choose the course of action feign Calais attack Normandy, and Germany will choose the course of action mobile inland with expected outcome valuations of $EV_{Allies} = 0.45$ and $EV_{Germany} = 0.55$, respectively.

**Operation Overlord/Fortress Europe
with Uncertainty**

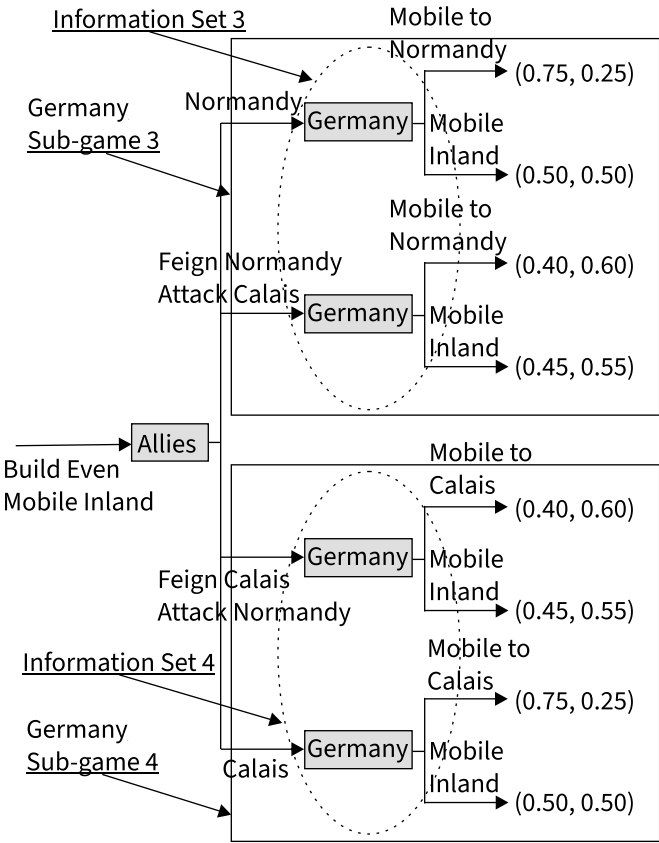


Figure 9.19: In this branch of the game, the Allied and German course-of-action pairs are linked to their respective expected outcome valuations (EV_{Germany}, EV_{Allies}). At this point in the game's solution, Germany has two sub-games, each coincidental with its two information sets.

Operation Overlord/Fortress Europe with Uncertainty

		Germany	
Sub-game 3		Mobile to Normandy	Mobile Inland
Allies	Attack Normandy	0.25 ↓	0.50 ↓ 0.50 ←
	Feign Normandy Attack Calais	0.60 ↓ 0.40 →	0.55 ← 0.45

Nash
Equilibrium

Figure 9.20: Because Germany's information set does not allow its leaders to know whether the Allies have chosen the course of action attack Calais or the course of action feign Calais attack Normandy, Sub-game 3 can be represented as this simultaneous-play game. Sub-game 3, as indicated by the arrows, has a discrete course-of-action Nash equilibrium in which the Allies choose the dominant course of action feign Normandy attack Calais and Germany chooses its course of action mobile inland.

Sub-game 4 from Figure 8.19 can be translated into a normal-form representation, as shown in Figure 9.21, for the same informational reasons as Sub-game 3. Inspecting Sub-game 4, as indicated by the arrows, we find a discrete course-of-action Nash equilibrium in which the Allies choose the dominant course of action to feign at Calais and attack Normandy. In turn, Germany chooses its course of action to hold its mobile forces inland, yielding an outcome valuation of 0.55 to the Allies and an outcome valuation of 0.45 to Germany in Sub-game 4, as shown in Figure 9.21. The outcome valuation symmetry between Sub-game 3 and Sub-game 4 drives the resolution to similar Nash equilibria in the two sub-games.

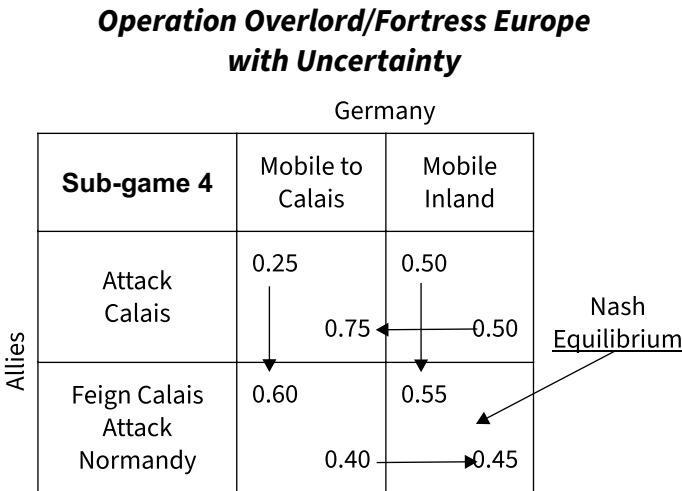


Figure 9.21: Because Germany’s information set does not allow it to know whether the Allies have chosen the course of action attack Calais or the course of action feign Calais attack Normandy, Sub-game 4 can be represented as this simultaneous-play game. Sub-game 4, as indicated by the arrows, has a discrete course-of-action Nash equilibrium in which the Allies choose the dominant course of action feign Normandy attack Calais and Germany chooses its course of action mobile inland.

Resolving up this branch of the game in which Germany has chosen the course of action to build evenly sized fixed coastal defenses and hold mobile forces inland, as represented in Figure 9.22, we find that the Allies are indifferent to being in Sub-game 3 or Sub-game 4. In Sub-game 3, the Allies have a dominant course of action to misdirect by choosing to feign at Normandy and attack Calais. In Sub-game 4, the Allies’ dominant course of action is again to misdirect, but this time by choosing to feign at Calais and attack at Normandy. Germany will respond to the Allied dominant course-of-action choice by choosing to hold its mobile forces inland in both sub-games. In either of the two Nash equilibria to the sub-games, the Allies have an expected outcome valuation of $EV_{Allies} = 0.45$, while Germany has an expected outcome valuation of $EV_{Germany} = 0.55$.

We now move to analyze the final branch of the game *Operation Overlord/Fortress Europe with Uncertainty* that has sub-games generated by information sets. This branch of the game is initiated by Germany choosing to build at Normandy with mobile forces inland. Germany, being unable to distinguish between the Allies' course-of-action choice to feign at Normandy and attack at Calais or to attack Normandy, could find itself in Information Set 5 corresponding to Sub-game 5. For similar reasons, Germany could find itself in Information Set 6 of Sub-game 6. This branch of the game is reproduced as Figure 9.23, initially presented as Figure 8.6.

We start our analysis with Sub-game 5 from Figure 9.23. Because of the information set restrictions, Germany does not know in which decision node it is located in Sub-game 5. This information asymmetry allows the extensive form of this sequential sub-game to be represented in the normal form of a simultaneous-play sub-game. This form of Sub-game 5 is presented in Figure 9.24.

In Sub-game 5, as indicated by the arrows, the Allies have a dominant discrete course of action, to feign at Normandy and attack at Calais, leading to a Nash equilibrium in which the Allies choose their dominant course of action. Germany responds by choosing to hold its mobile forces inland. In this Nash equilibrium, the Allies have an expected outcome valuation of $EV_{\text{Allies}} = 0.55$, and Germany has an expected outcome valuation of $EV_{\text{Germany}} = 0.45$.

Sub-game 6 in this branch of the game has the same informational characteristics encountered in Sub-game 5. Given Germany's inability to know at which decision it is located in Sub-game 6, the extensive form of the sequential-play Sub-game 6 can be represented in the normal form of a simultaneous-play Sub-game 6. The simultaneous-play form of Sub-game 6 is presented in Figure 9.25.

**Operation Overlord/Fortress Europe
with Uncertainty**

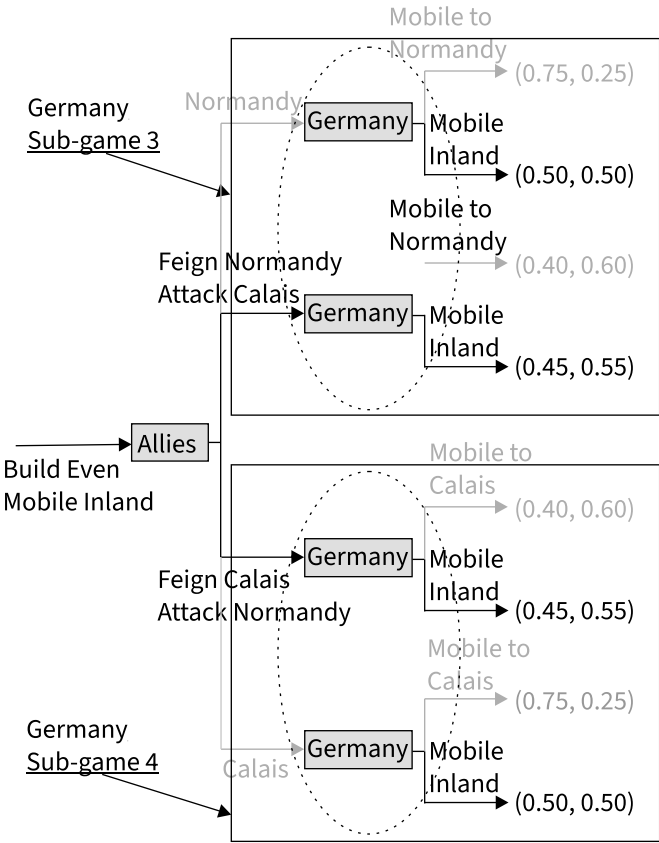


Figure 9.22: In this branch of the game, the Allies are indifferent to the choice between Sub-game 3 and Sub-game 4. The Allies' dominant course of action is to misdirect by feigning Normandy in Sub-game 3 or feigning Calais in Sub-game 4. Germany chooses the course of action mobile inland. Both Nash equilibria result in the Allies' expected outcome valuation of 0.45 and Germany's expected outcome valuation of 0.55.

Operation Overlord/Fortress Europe with Uncertainty

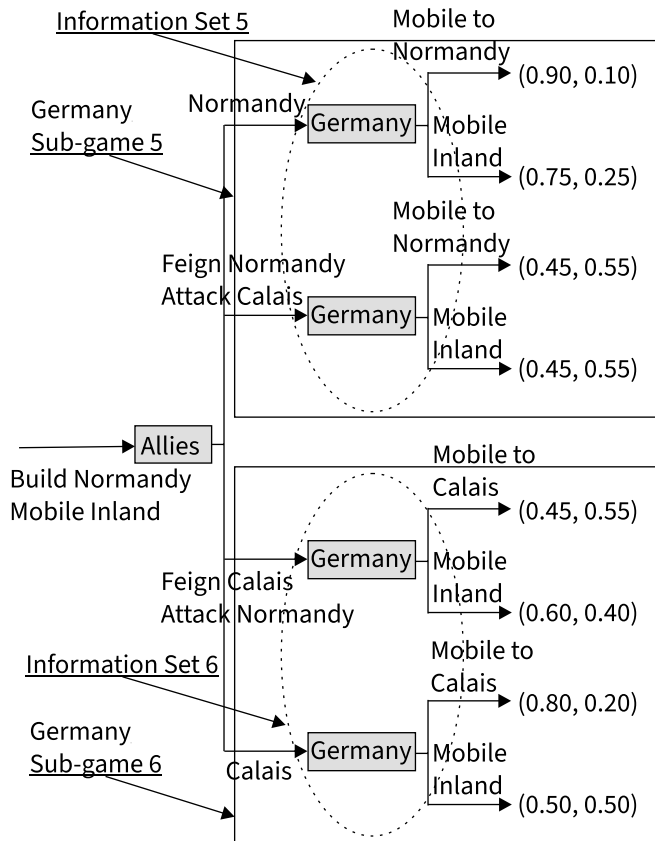


Figure 9.23: In this branch of the game, the Allied and German course-of-action pairs are linked to their respective expected outcome valuations (EV_{Germany} , EV_{Allies}). At this point in the game's solution, Germany has two sub-games, each coincidental with its two information sets.

**Operation Overlord/Fortress Europe
with Uncertainty**

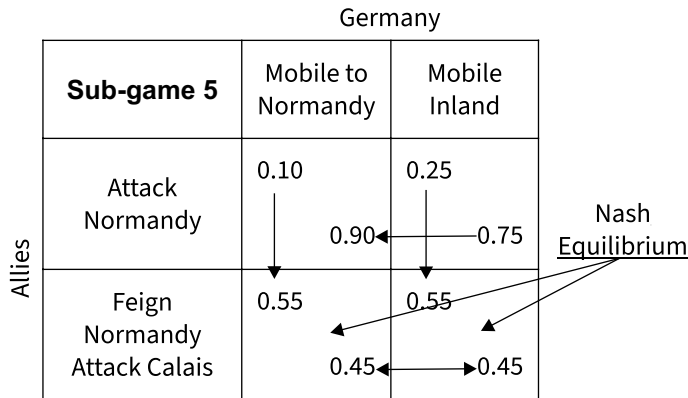


Figure 9.24: In this simultaneous-play representation of Sub-game 5, as indicated by the arrows, the Allies have a dominant discrete course of action, feign Normandy attack Calais, leading to a Nash equilibrium in which the Allies choose the dominant course of action feign Normandy attack Calais, and Germany chooses its course of action mobile inland.

As seen in Figure 9.25, there are no discrete course-of-action equilibria. Thus, the analysis of Sub-game 6 turns to finding a mixed course-of-action equilibrium. The Allies will select a probability ρ_{AC} that they will choose their course of action to attack at Calais and thus the probability $(1-\rho_{AC})$ that they will choose their course of action to feign at Calais and attack Normandy. Germany, knowing that the Allies have a probability distribution across their course-of-action choices, will calculate its expected outcome valuations conditional upon the Allies' probability distribution.

Germany's expected valuation to the course of action to deploy its mobile forces to Normandy, given the Allies' choice of ρ_{AC} , is $EV(\text{Mobile to Calais})_{\text{Germany}} = 0.45 + 0.35 \times \rho_{AC}$. Similarly, Germany's expected valuation for holding its mobile forces inland, given the Allies' choice of ρ_{AN} , is $EV(\text{Mobile Inland})_{\text{Germany}} = 0.60 - 0.10 \times \rho_{AC}$. The calculation of Germany's expected valuations, $EV(\text{Mobile to Calais})_{\text{Germany}}$ and $EV(\text{Mobile Inland})_{\text{Germany}}$, are developed in Figure 9.26.

**Operation Overlord/Fortress Europe
with Uncertainty**

		Germany	
		Mobile to Calais	Mobile Inland
Allies	Sub-game 6		
	Attack Calais	0.20 ↓	0.50 ↑ ← 0.50
	Feign Calais Attack Normandy	0.45 ↓ ← 0.55	0.40 ← 0.60

Figure 9.25: In this simultaneous-play representation of Sub-game 6, as indicated by the arrows, there are no discrete course-of-action Nash equilibria. There is no cell in which the Allies and Germany have mutual best responses.

The Allies will select their probability of choosing their course of action to attack at Calais ρ_{AC} in a way that prevents Germany from systematically exploiting the Allies' choice. When Germany is indifferent to choosing between its course of action to deploy mobile forces to Calais and to hold mobile forces inland, the Allies have chosen an exploitation-proof ρ_{AC} . This happens when

$$\begin{aligned} \text{EV}(\text{Mobile to Calais})_{\text{Germany}} &= \text{EV}(\text{Mobile Inland})_{\text{Germany}}, \\ 0.55 + 0.25 \times \rho_{AC} &= 0.60 - 0.10 \times \rho_{AC}, \text{ and} \\ 0.35 \times \rho_{AN} &= 0.05 \text{ or } \rho_{AC} = \frac{1}{7}. \end{aligned}$$

Thus, the Allies will mix up their course-of-action choices such that they choose attack Calais with probability $\rho_{AC} = \frac{1}{7}$.

Germany’s Expected Valuations Given p_{AC}

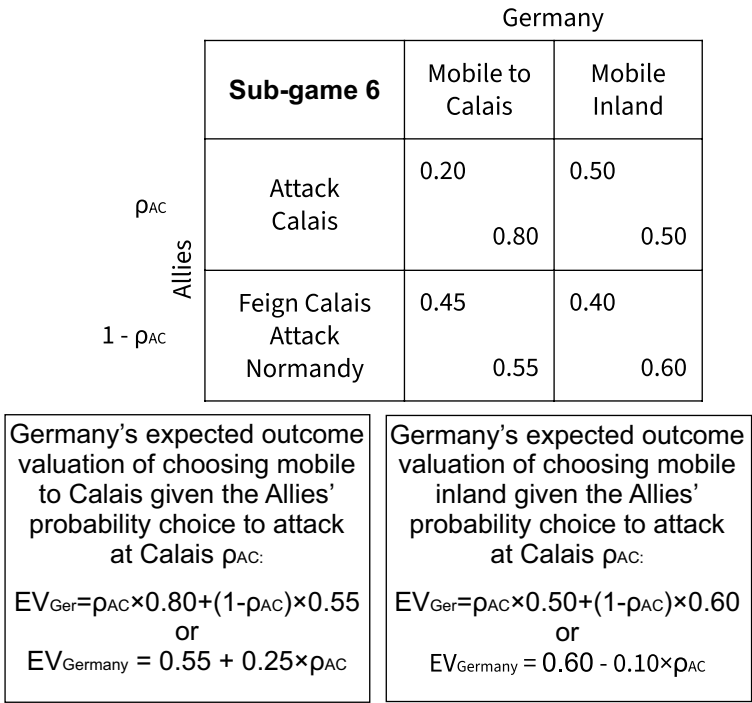


Figure 9.26: The Allies choose the probability p_{AC} , such that Germany is indifferent to the choice to deploy mobile forces to Calais or to hold mobile forces inland.

Next, the Germans will select a probability ϕ_{MC} that they choose their course of action to deploy mobile forces to Calais and thus the probability $(1-\phi_{MC})$ that they choose their course of action to hold mobile forces inland. The Allies, knowing that Germany has a probability distribution across its course-of-action choices, will calculate the Allied expected outcome valuations conditional upon Germany’s probability distribution. This is developed in Figure 9.27.

The Allies’ expected valuation for their course of action to attack Calais, given Germany’s choice of ϕ_{MC} , is

$$EV(\text{Attack Calais})_{Allies} = 0.50 + 0.30 \times \phi_{MC}.$$

Similarly, the Allies' expected valuation for their course of action to feign at Calais and attack Normandy, given Germany's choice of ϕ_{MC} , is

$$EV(\text{Feign Calais Attack Normandy})_{\text{Allies}} = 0.40 - 0.05 \times \phi_{MC}.$$

The calculations of the Allies' expected valuations, $EV(\text{Attack Calais})_{\text{Allies}}$ and $EV(\text{Feign Calais Attack Normandy})_{\text{Allies}}$, are developed in Figure 9.27.

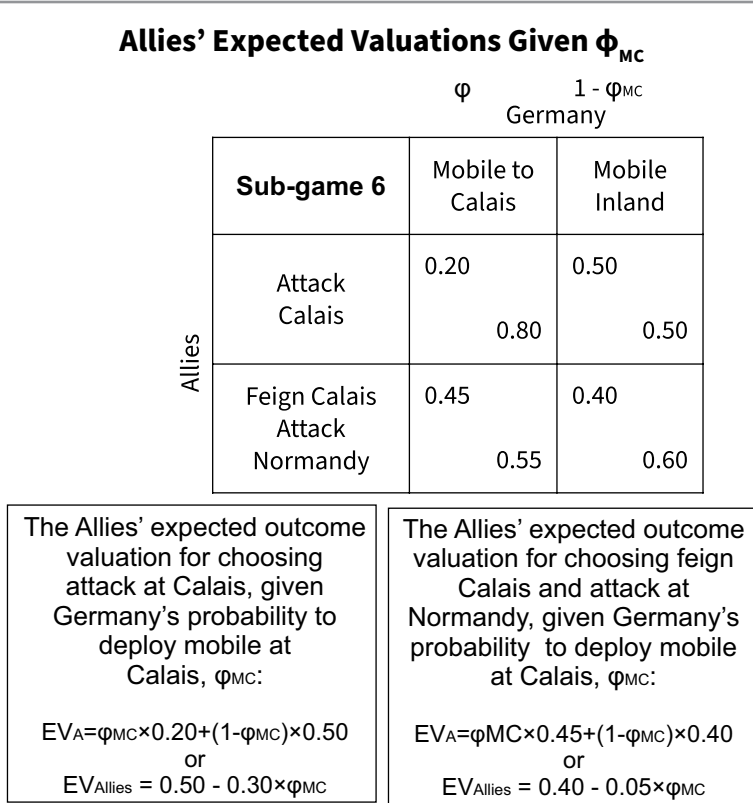


Figure 9.27: Germany chooses the probability ϕ_{MC} , such that the Allies are indifferent to the choice to attack at Calais or to feign to attack Calais and attack at Normandy.

The Germans will select a probability of choosing their course of action ϕ_{MC} in a way that prevents the Allies from systematically exploiting Germany's choice. While the Allies are indifferent to choosing between the course of action to attack Calais and to feign at Normandy and attack Calais, they have chosen an exploitation-proof ϕ_{MC} . This happens when

$$\begin{aligned} EV(\text{Attack Calais})_{\text{Allies}} &= EV(\text{Feign Normandy Attack Calais})_{\text{Allies}}, \\ 0.50 + 0.30 \times \phi_{MC} &= 0.40 - 0.05 \times \phi_{MC}, \text{ or} \\ 0.35 \times \phi_{MC} &= 0.10 \text{ or } \phi_{MC} = \frac{2}{7}. \end{aligned}$$

Thus, Germany will vary its course-of-action choices such that they choose mobile to Calais with probability $\phi_{MC} = \frac{2}{7}$.

At the mixed-strategy equilibrium ($\phi = \frac{2}{7}$ $\rho = \frac{1}{7}$), Germany's expected outcome valuation in Sub-game 1 is

$$\begin{aligned} EV(\text{Mobile to Calais})_{\text{Germany}} &= 0.80 \times \frac{1}{7} + 0.55 \times \frac{6}{7} = 0.59 \\ EV(\text{Mobile Inland})_{\text{Germany}} &= 0.50 \times \frac{1}{7} + 0.60 \times \frac{6}{7} = 0.59, \end{aligned}$$

regardless of its course-of-action choice. Similarly, the Allies' expected outcome valuation in Sub-game 1 is

$$\begin{aligned} EV(\text{Attack Calais})_{\text{Allies}} &= 0.2 \times \frac{2}{7} + 0.5 \times \frac{5}{7} = 0.41 \\ EV(\text{feign Cal attack Norm})_{\text{Allies}} &= 0.45 \times \frac{2}{7} + 0.40 \times \frac{5}{7} = 0.41, \end{aligned}$$

regardless of their course-of-action choice. So Sub-game 6 of *Operation Overlord/Fortress Europe with Uncertainty* resolves to the mixed-strategy equilibrium in which the Allies choose to attack Calais with the probability of $\rho = \frac{1}{7}$ and an expected outcome valuation of $EV_{\text{Allies}} = 0.41$. Using the same process, Germany chooses to deploy mobile forces to Normandy with the probability of $\phi = \frac{2}{7}$ and an expected outcome valuation of $EV_{\text{Germany}} = 0.59$.

Having solved Sub-games 5 and 6, we can generate Figure 9.28 by substituting those solutions into the branch of the game where Germany's initial course-of-action choice is to build fixed coastal defenses at Normandy with mobile forces inland, as represented in Figure 9.23. Looking at Sub-game 5 and knowing the Allies have a dominant course of action to feign at Normandy and attack Calais, Germany is indifferent to choosing between deploying mobile forces to Normandy and holding mobile forces inland. This yields the two

equilibria to Sub-game 5, in which the Allies choose to feign at Normandy and attack Calais, and Germany responds with either of its two courses of action. Both equilibria in Sub-game 5 yield expected outcome valuations to Germany of $EV_{\text{Germany}} = 0.45$ and expected outcome valuations to the Allies of $EV_{\text{Allies}} = 0.55$. Looking at Sub-game 6, Germany and the Allies have a mixed course-of-action equilibrium in which the Allies will choose the course of action to attack Calais with probability $p_{AC} = 1/7$ and Germany will choose the course of action to deploy mobile forces to Calais with probability $\phi_{MC} = 2/7$. At this probability mix, the Allies have an expected outcome valuation of $EV_{\text{Allies}} = 0.41$ and Germany has an expected outcome valuation of $EV_{\text{Germany}} = 0.55$. Thus, if Germany chooses an initial course of action to build at Normandy with its mobile forces inland, a rational Allied response in Sub-game 5 would be to feign at Normandy and attack Calais. Germany would then be indifferent to choosing between its two course-of-action choices. As indicated in Figure 9.28, this branch of the game would resolve to the outcome highlighted in Sub-game 5.

The Sub-games of *Operation Overlord/Fortress Europe with Uncertainty* have all been resolved up to the point where Germany chooses its initial course of action. Germany can now look at the game tree, as presented in Figure 9.29, and make its choice, knowing how the game will play out if the Allies make their own course-of-action choices to give them the highest expected outcome valuation. As observed in Chapter VII, the actual course of action chosen by Germany—to build at Calais with mobile forces inland—did not set Germany on a game path that would be most beneficial to its goals, with expected outcome valuations of $EV_{\text{Germany}} = 0.45$ and $EV_{\text{Allies}} = 0.55$. The same two equilibria from the case study in Chapter VII emerge in this mixed-course-of-action-equilibria version of the game. This equilibrium path starts with Germany choosing the course of action to build evenly sized fixed coastal defenses while deploying mobile forces to the coast. The Allies would then be indifferent to choosing between the choices to attack Normandy and attack Calais. These equilibria paths would then yield expected outcome valuations of $EV_{\text{Germany}} = 0.75$ and $EV_{\text{Allies}} = 0.25$.

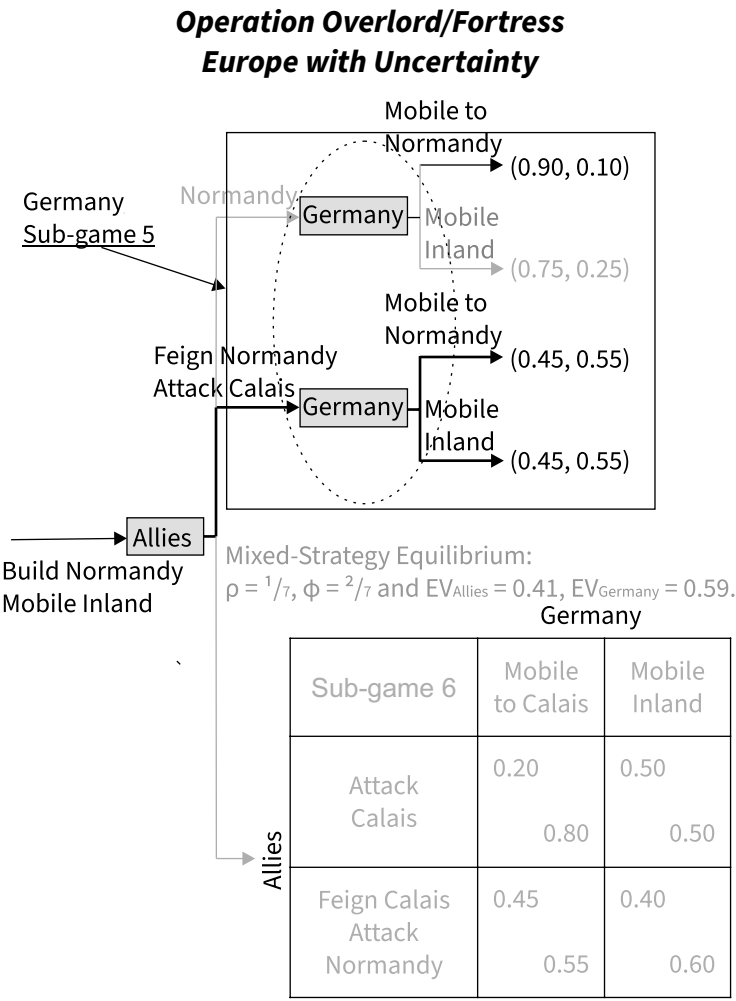


Figure 9.28: In this branch of the game, the Allies would choose to be in Sub-game 5 by choosing their course of action feign Normandy attack Calais, resulting in expected outcome valuations of $EV_{Allies} = 0.55$ and $EV_{Germany} = 0.45$.

Operation Overlord/Fortress Europe with Uncertainty

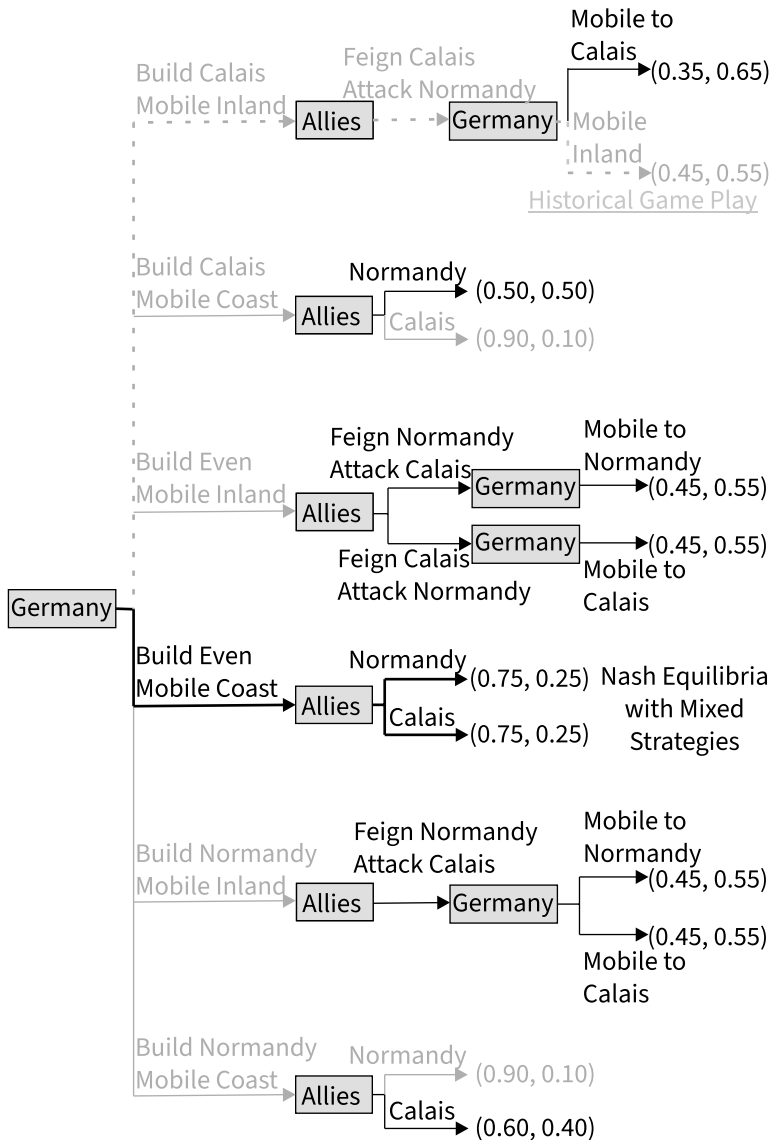


Figure 9.29: The game *Operation Overlord/Fortress Europe with Uncertainty* with the expected outcome valuations (EV_{Germany} , EV_{Allies}) resolved for all the branches of the game. The Nash equilibrium with mixed strategies is the dark highlighted choice path.

As mentioned earlier, Germany's initial course of action to build fixed defenses evenly and deploy mobile forces to the coast was German Field Marshall Rommel's recommended strategy for the German defense of Europe. This initial course of action was not preferred by his superior, Field Marshall Rundstedt, who preferred moving the Panzer divisions inland for a counteroffensive. Rommel believed that Panzer divisions in the rear would not be able to move up to the fight fast enough to be an effective counteroffensive, given the Allies' massive air superiority.³ Regardless of his protests, Rundstedt's course of action to build fixed coastal defenses at Calais with mobile forces inland won approval of the German command all the way up to Hitler.

Rommel's battlefield intuition, coupled with his Panzer experience against Allied airpower in Africa, were key to his understanding of a successful defense of Europe. But his subordinate-officer intuition, particularly when dealing with an old Prussian superior, held little or no strategic credibility. Even in the modern armed forces, intuition flowing up the chain of command from subordinate officer to superior officer is often not sufficient. A well-argued case for the subordinate officer's intuition overriding the superior officer's decision prerogative is required. It is in this type of situation—with a Rommel arguing up to a Rundstedt—that a game-theoretic structure can be used as a device to legitimize the efficacy of an intuitive course of action from an opinion to a deductively reasoned initiative.

CHAPTER X

CONCLUSION:
DECIDING WELL IN
STRATEGIC-CHOICE
ENVIRONMENTS

*Thus one who excels at warfare first establishes himself
in a position where he cannot be defeated while not
losing [any opportunity] to defeat the enemy.*

*For this reason, the victorious army first
realizes the conditions for victory, and then
seeks to engage the battle. The vanquished
army fights first, and then seeks victory.*

SUN TZU, *ART OF WAR*

Game Theory and the Art of War is a primer for the intersection between the mathematical world of decision-making in an environment where there are competitive actors with differing agendas and the military world of tactical and strategic planning, operation, and analysis. This book is based on a simple but demanding premise: that warfare, like many of the most consequential human endeavors, is fundamentally an interactive decision problem. Outcomes are not determined in isolation with brilliance, superior resources, or even doctrinal correctness, but by the interaction of choices made by thinking adversaries under conditions of uncertainty. Throughout these chapters, we have argued that game theory—properly understood and carefully applied—offers leaders a disciplined way to think about that interaction. It does not replace judgment, experience, or intuition; rather, it sharpens them.

The central purpose of this book is to reposition game theory not as an abstract mathematical enterprise but as a *practical discipline of strategic reasoning*. Game theory is the thought arena in which competitors' choices are analyzed as they interact and unfold into outcomes. By stripping away unnecessary formalism while preserving analytical rigor, we demonstrate how game theory can be used to clarify strategic-choice environments, identify credible courses of action, and explain why reasonable leaders sometimes arrive at profoundly different decisions. In doing so, we seek to make game theory accessible to practitioners without diminishing its explanatory power.

Across the chapters, the game-theoretic thought process develops cumulatively. We begin by defining what it means to “make the game”: to choose an appropriate level of abstraction, identify relevant players (including nonstrategic actors, such as Nature), specify feasible courses of action, and assign meaningful valuations to outcomes.

This foundational work is not ancillary to strategy; it *is* strategy. As the parables of maps and models illustrate, the utility of any analytical framework depends on its appropriateness to the decision problem at hand. Too much detail obscures what matters; too little distorts reality.

From this foundation, we move to “playing the game”—that is, to understanding how rational, empathetic actors behave once the structure of the game is specified. We introduce concepts such as rational choice, dominant strategies, Nash equilibrium, and sub-game perfect equilibrium not as ends in themselves but as tools for anticipating behavior and evaluating choices. Through examples ranging from siege warfare to amphibious invasion, we show how backward induction and expected-value reasoning allow leaders to reason forward through uncertainty and then back toward sound decisions.

The analysis of *Operation Overlord/Fortress Europe* illustrates the full power of this approach. By modeling the strategic interaction between German and Allied planners, incorporating probabilistic outcomes through Nature, and solving for sub-game perfect equilibria, we demonstrate that game theory can generate insights that align with historical debates while also reframing them. Crucially, the model does not “solve” history in a deterministic sense. Instead, it clarifies why different strategic visions—those associated with Rommel and von Rundstedt—can each be internally rational given differing beliefs about the same environment.

This insight points to one of the book’s most important contributions: the treatment of *beliefs and scenarios* as central objects of strategic analysis. Strategic disagreement is often mischaracterized as incompetence, irrationality, or bad faith. Game theory offers a more charitable—and more useful—interpretation. When leaders disagree, they may be playing different games, informed by different experiences, assumptions, and probability assessments. By making those assumptions explicit, game theory transforms disagreement from a clash of personalities into a contest of models that can be examined, debated, and refined.

When leaders disagree about the probability distribution that their competitors will apply to strategy mixes in a game with uncertainty, the leaders can assess whether their beliefs are rational. For beliefs to be rational, they must be consistent and logical about the opponents' rational actions given the game's structure. Mixed strategy equilibrium is a powerful tool to eliminate irrational beliefs and scenarios, allowing leaders to assess and prune away irrelevant game formulations.

In this sense, game theory serves not only as an evaluative tool but also as a communicative one. It provides a common language for articulating why a course of action appears attractive, what assumptions sustain that judgment, and where points of genuine uncertainty lie. For military organizations—where persuasion, coordination, and shared understanding are as vital as command authority—this function is indispensable.

The broader significance of this approach extends beyond military history or battlefield decision-making. Game theory has grown, from a mathematical curiosity introduced by mathematician John von Neumann and economist Oskar Morgenstern in their 1944 book *Theory of Games and Economic Behavior*, to a powerful analytical tool for strategic interaction between participants with competing interests. Though this book uses simple games and well-known historical battles as tools for developing and illustrating the game-theoretic thought process for the reader, these tools are useful in analyzing and extracting deeper understanding of today's major international strategic issues and those moving forward. Examples of current strategic interactions include: the Great Power competition between the United States and China in the Pacific rim, as well as between NATO and Russia in Europe; regional instability in Africa and the Middle East; drug cartel remediation in South and Central America; Economic Nationalism; illicit migration; and trade negotiations and policies. Any domain characterized by strategic interaction—politics, diplomacy, economics, organizational leadership—shares the same underlying structure. What distinguishes military applications is the severity of the stakes and the irreversibility of many outcomes. In such

environments, the cost of poor reasoning is measured not only in lost opportunities but also in lost lives, legitimacy, and national standing. The discipline of thinking that game theory encourages is therefore not merely useful; it is ethically consequential.

At the same time, we have been careful to acknowledge what game theory cannot do. Models are abstractions, not mirrors of reality. They simplify, omit, and stylize. They depend on assumptions that may be wrong or incomplete. No equilibrium concept can capture the full richness of human motivation, organizational dysfunction, or chance. Recognizing these limitations is not a weakness of the approach but a condition of its responsible use. Game theory disciplines thinking; it does not absolve decision-makers of responsibility for judgment.

Several implications follow from this recognition. First, the value of game theory lies less in the precision of its predictions than in the clarity of its reasoning. A well-constructed model can be useful even if its quantitative estimates are imprecise, so long as it illuminates strategic structure and highlights critical dependencies. Second, disagreement among models should be expected and welcomed. Competing games reflect and provide form to competing understandings of reality, and progress often comes from examining why those understandings diverge.

Finally, this approach invites further work. Future research might integrate richer behavioral assumptions, explore learning and adaptation over repeated interactions, or combine game-theoretic reasoning with empirical simulation and historical data. For practitioners, the next step is not more mathematics but more practice: applying these tools to new cases, refining assumptions, and cultivating the habit of explicit strategic reasoning. As you mature in your game-theoretic analysis, you will want to dig deeper into the subject by consulting works such as *Games of Strategy* by Avinash K. Dixit, Susan Skeath, and David McAdams, published by W. W. Norton & Company; *Two Person Game Theory* by Anatol Rapoport, published by the University of Michigan Press; and *Game Theory: Analysis of Conflict* by Roger Myerson, published by Harvard University Press.

In an age defined by complexity, speed, and uncertainty, the greatest advantage may lie not in superior information or technology, but in the ability to *decide well*. Game theory, as developed in this book, offers no guarantees of victory. What it offers instead is something more enduring: a way to see strategic situations more clearly, to anticipate interaction rather than react to it, and to act with deliberation when the stakes are highest.

That, ultimately, is the art of war—and the enduring promise of game theory as a guide to strategic judgment.



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27. At Gettysburg, each Union corps had approximately 11,000 soldiers and each Confederate corps had approximately 22,000.
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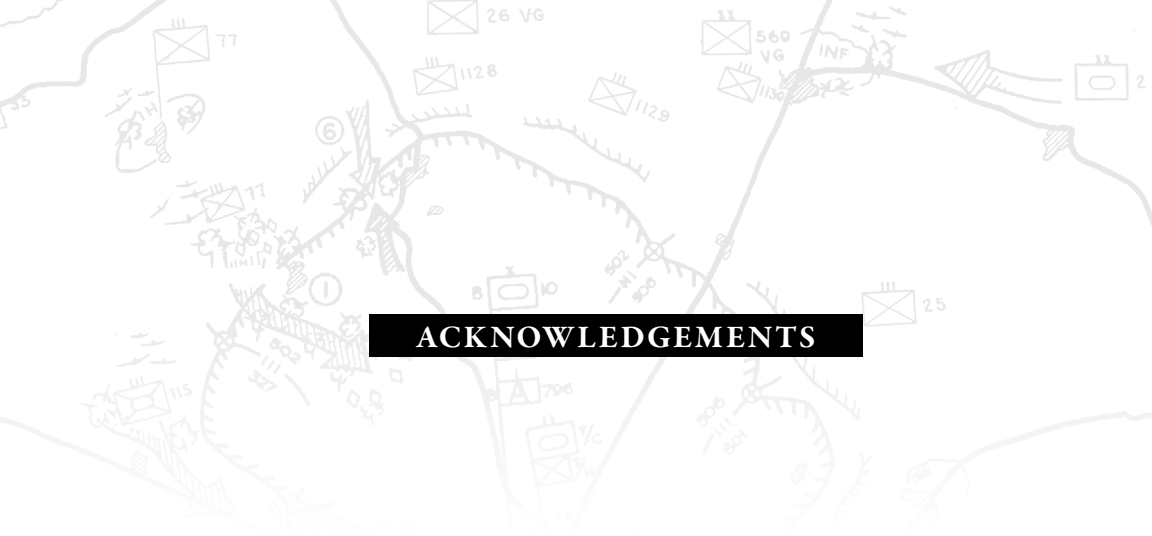
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ACKNOWLEDGEMENTS

DEAN DUDLEY:

First and foremost, I want to thank Sarah D for her support and encouragement throughout this project. Without her edits, common sense, and persistence it would never have come to fruition. I can never give her enough credit for the inspiration and love she gave me. In a close second comes Tony Martin, my co-author. His patience and hard work were critical to this book's production.

I want to thank, at a formative level, James Walker, Elinor Ostrom, and Roy Gardner, valued colleagues and professors who groomed me in the foundations of experimental economics, the theory of political economy, and game theory. In addition to these three mentors, I would like to thank Roger Myerson who sacrificed hours of his time in discussions about some of the deep particulars of the game-theoretical constructions that Tony and I employed in this book. I would also like to thank Roger for seeing merit in and encouraging a mathematically light game theory primer for strategic military practitioners. Finally, I want to thank Chip Frank for very helpful feedback on earlier portions of this work presented in various sessions at the Western Economics Association's conferences.

Additional thanks go to the three blind reviewers of the initial manuscript for their helpful guidance, corrections, and ideas, which resulted in the composition of a much better product than Tony and I would have been able to generate without their selfless service.

TONY MARTIN:

I wish to thank above all others my wife, Valerie. Her patience, strength, and unwavering partnership across the long seasons of military life have been a steady anchor through every challenge and change. Every soldier should be so fortunate as to walk through both peace and war with such a companion at their side.

I am deeply grateful to Dr. Emerson Niou, whose guidance during my doctoral years and whose intellectual rigor shaped the habits of analysis that became this book's foundation. His mentorship not only deepened my understanding of game theory but also inspired me to translate those insights into practical tools for leaders at West Point and beyond.

Finally, I extend my profound respect and thanks to the soldiers with whom I have served. They taught me lessons that reach far beyond tactics and operations—lessons about discipline, humility, courage, and purpose. From them I learned that the art of readiness is not about seeking conflict, but about preparing so thoroughly that, should conflict come, we are ready to preserve life and uphold the values we defend.



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